Assignment 4

November 10, 2023

1. For the canonical ensemble,
   (a) Show that the isothermal compressibility κ_T can be expressed as $\kappa_T = -\frac{1}{Vk_BT}(\frac{\partial^2}{\partial V^2}lnZ)^{-1}$, where Z is the canonical partition function.
   (b) Show that the variance in energy ($\langle \Delta E^2 \rangle$) is proportional to the heat capacity at constant volume, C_V.
2. (a) Derive the expression for grand canonical partition function $Z_g(\mu, V, T)$ in the grand canonical ensemble. Show its relation with the grand canonical potential, pV.
   (b) Using the grand canonical partition function, show that the entropy (S), average number of particles (N) and the average pressure (p) are given by,

$$S = k_B lnZ_g + k_B T(\frac{\partial lnZ_G}{\partial T})_{V,\mu}$$

$$N = k_B T(\frac{\partial lnZ_G}{\partial \mu})_{V,T}$$

$$p = k_B T\frac{lnZ_g}{V}$$

   (c) Consider an ideal gas in the grand canonical ensemble. Derive expressions for the average number of particles and internal energy in terms of temperature, chemical potential, and volume.
   (d) Show that in the grand canonical ensemble,

$$\frac{\sigma_N}{\overline{N}} = (\frac{k_B T\kappa}{V})^{\frac{1}{2}}$$

   where σ_N is the standard deviation in particle number and κ is the isothermal compressibility.
3. The average occupancy in the Bose-Einstein statistics is given as

$$\overline{n_i} = \frac{1}{e^{-\beta(\mu-\epsilon_i)} \pm 1}$$

   (a) The lowest energy state, ϵ_0 is an upper limit for the chemical potential, μ. Why?
   (b) What happens to the occupation number when the chemical potential approaches the lowest energy level, ϵ_0 from below. Explain.