CHM 428 Assignment 2

February 17, 2024

Due on 25^{th} Feb., 2024.

1. The operator for infinitesimal translations along the x-axis is given by the expression $\hat{T}(dx) = \hat{1} + i\frac{\hat{p}_x}{\hbar}dx$, where $\hat{p}_x$ is the x component of the momentum operator. Show that $\hat{T}$ satisfies the following up to first order in dx:
   (a) $\hat{T}(dx)\hat{T}^\dagger(dx) = \hat{T}^\dagger(dx)\hat{T}(dx) = \hat{1}$
   (b) $\hat{T}(dx + dx') = \hat{T}(dx)\hat{T}(dx')$
   (c) $\hat{T}^\dagger(dx) = \hat{T}(-dx)$

2. From the definition of the operator above show that $\langle x|\hat{T}(dx)|\Psi\rangle = \Psi(x + dx)$ to first order in dx.

3. Express the following operators in momentum basis kets $\{|p_x\rangle\}$.:
   (a) $\hat{x}$
   (b) $\hat{p}_x$
   (c) $\exp\left(-\hat{p}_x^2/2m\right)$ (where m is the mass of a particle).

4. The kinetic energy operator of a particle (mass m) confined to move along the x-direction is given by $\hat{T} = \frac{\hat{p}^2}{2m}$. Show that in position representation the operator is $-\frac{\hbar^2}{2m}\frac{d^2}{d^2x}$.

5. The potential energy operator of a particle (mass m) confined to move along the x-direction is denoted by $V(\hat{x})$. Is it local in space? Determine the operator in momentum representation. Is it local in momentum?

6. Show that the following relations hold

   (a) $[\hat{x}, \hat{p}_x^m] = i\hbar\hat{p}_x^{m-1}$

   (b) $[\hat{x}, f(\hat{p}_x)] = i\hbar f'(\hat{p}_x)$

   (c) $[g(\hat{x}), \hat{p}_x] = i\hbar g'(\hat{x})$

   where f and g are two continuous functions and the primed quantities are their corresponding derivatives. <u>Hint</u>: After solving the first part, use a Taylor series expansion for f anf g about 0 to solve the next two parts.

7. The Hamiltonian operator of a particle moving in 1-dimension is given as $\hat{H} = \hat{T} + V(\hat{x})$, where $\hat{T}$ is the kinetic energy operator (see above) and $V(\hat{x})$ is the potential energy operature defined as a function of $\hat{x}$. Compute the commutator $\hat{C} = \left[\hat{x}, \hat{H}\right]$ and show that the expectation value of $\left[\hat{x}, \hat{C}\right]$ is a constant for any arbitrary (normalized) state.

8. The position representation wavefunction of a 1-dimensional particle in a certain energy eigenstate is given by $\psi(x) = N\exp(-\frac{x^2}{2\sigma^2})$, where N is a normalization constant. Show that the momentum representation of the same state also has the same functional form (Gaussian) and determine the corresponding σ. Show that the momentum-space and position-space σ's are inversely related. What can you say about the uncertainty of locating the particle in space from the nature of ψ?