CHM 428 Assignment 2

September 5, 2025

Due on 15^{th} Sep., 2025 in class.

1. (a) Prove the generalized uncertainty principle

$$\left\langle \Delta\hat{\alpha}^2 \right\rangle \left\langle \Delta\hat{\beta}^2 \right\rangle \geq \frac{1}{4} \left| \left\langle \left[\hat{\alpha}, \hat{\beta} \right] \right\rangle \right|^2$$

where $\hat{\alpha}, \hat{\beta}$ are two observables.

(b) Derive an uncertainty relation for the components of orbital angular momentum $\vec{L}$, given that $\left[\hat{L}_i, \hat{L}_j\right] = \epsilon_{ijk} \mathrm{i}\hbar \hat{L}_k$, where i, j, k run over the cartersian coordinates x, y, z and ε_{ijk} is the Levi-Cevita symbol

$$\varepsilon_{ijk} = \begin{cases} +1 & \text{if } (i,j,k) \text{ is an even permutation of } (x,y,z) \\ -1 & \text{if } (i,j,k) \text{ is an odd permutation of } (x,y,z) \\ 0 & \text{if any two indices are equal} \end{cases}$$

2. Show that a Hermitian operator $\hat{\alpha}$, when represented in an orthonormal basis $\{|\lambda_i\rangle\}$, corresponds to a Hermitian matrix $\mathbf{A}$, such that $\mathbf{A}_{ij} = \langle\lambda_i|\hat{\alpha}|\lambda_j\rangle$ and $\mathbf{A}_{ij} = (\mathbf{A}_{ji})^*$.

3. The operator $\hat{\eta}$ that converts one orthonormal representation $(\{|\lambda_i\rangle\})$ into another $(\{|\xi_i\rangle\})$ as $|\xi_i\rangle = \hat{\eta}|\lambda_i\rangle$. Show that $\hat{\eta}$ is represented in either basis by a unitary matrix $\mathbf{U}$, such that $\mathbf{U}^\dagger\mathbf{U} = \mathbf{U}\mathbf{U}^\dagger = I$, where I is the identity matrix.

4. Prove the following properties of the Dirac delta function $\delta(x)$, given its basic definition as

$$\int_{-\infty}^{\infty} \delta(x)dx = 1 \tag{1}$$

$$\int_{-\infty}^{\infty} \delta(x)f(x-a)dx = f(a) \tag{2}$$

   (a) $\delta(x) = \delta(-x)$
   (b) $\delta(ax) = \frac{1}{|a|}\delta(x)$
   (c) $\int_{-\infty}^{\infty} \delta'(x)f(x-a)dx = f'(a)$
   (d) $\delta(g(x)) = \sum_{i=1}^{N_{roots}} \frac{\delta(x-x_i)}{|g'(x_i)|}$, where the sum is over the number of roots of the real function $g(x)$.

5. Given the basic commutator, $[\hat{x}, \hat{p}_x] = \mathrm{i}\hbar$, show that the following relations hold

   (a) $[\hat{x}, \hat{p}_x^m] = \mathrm{i}\hbar\hat{p}_x^{m-1}$
   (b) $[\hat{x}, f(\hat{p}_x)] = \mathrm{i}\hbar f'(\hat{p}_x)$
   (c) $[g(\hat{x}), \hat{p}_x] = \mathrm{i}\hbar g'(\hat{x})$

   where f and g are two continuous functions and the primed quantities are their corresponding derivatives. Hint: After solving the first part, use a Taylor series expansion for f anf g about 0 to solve the next two parts.

6. The Hamiltonian operator of a particle moving in 1-dimension is given as $\hat{H} = \hat{T} + V(\hat{x})$, where $\hat{T}$ is the kinetic energy operator (see above) and $V(\hat{x})$ is the potential energy operature defined as a function of $\hat{x}$. Compute the commutator $\hat{C} = \left[\hat{x}, \hat{H}\right]$ and show that the expectation value of $\left[\hat{x}, \hat{C}\right]$ is a constant for any arbitrary (normalized) state.

7. The position representation wavefunction of a 1-dimensional particle in a certain energy eigenstate is given by $\psi(x) = N\exp\left(-\frac{x^2}{2\sigma^2}\right)$, where N is a normalization constant. Show that the momentum representation of the same state also has the same functional form (Gaussian) and determine the corresponding σ. Show that the momentum-space and position-space σ's are inversely related. What can you say about the uncertainty of locating the particle in space from the nature of ψ? Hint: $\int\limits_{-\infty}^{\infty} dx \; \exp\left(-ax^2\right) \mathrm{e}^{\mathrm{i}2\pi kx} dx = \sqrt{\frac{\pi}{a}}\exp\left(-\frac{\pi^2 k^2}{a}\right)$