25/8/2025 Day 9. Functions of observables and expectation values

Postulate 4: (Born's postulate).

The probability that a measurement of $\hat{\alpha}$ on the state $|\Psi\rangle$ would yield a value α' is proportional to

$$|\langle\alpha'|\Psi\rangle|^2$$

If $|\Psi\rangle$ is normalized then

$$\boxed{P_{\alpha'} \equiv |\langle\alpha'|\Psi\rangle|^2} \quad \text{— (1)}$$

As per postulate 3 we should be able to write any state $|\Psi\rangle$ in terms of the eigenstates of any observable $\hat{\alpha}$.

i.e. $$|\Psi\rangle = \underbrace{\left(\sum_{\alpha'} |\alpha'\rangle\langle\alpha'|\right)}|\Psi\rangle \quad \text{— (2)}$$

$$\Rightarrow \sum_{\alpha'} |\alpha'\rangle\langle\alpha'| = \hat{1} \quad \text{— (2)}$$

What if the observable is associated with continuous spectrum of eigenvalues.

e.g. $\hat{x}$: $\hat{x}|x'\rangle = x'|x'\rangle, x' \in \mathbb{R}$

The postulate intuitively suggests that in such a case of continuous eigenvalues (say, ξ') we should be able to write,

$$|\Psi\rangle = \int d\xi' \, |\xi'\rangle \quad \text{--- (4)}$$

($|\xi'\rangle$ not necessarily normalized)

If $|\Psi\rangle$ is normalized then,

$\langle\Psi|\Psi\rangle = 1 \Rightarrow$

$$\int d\xi' \int d\xi'' \, \langle\xi'|\xi''\rangle = 1 \quad \text{--- (5)}$$

But, for $\xi' \neq \xi''$ we must have $\langle\xi'|\xi''\rangle = 0$ — (6)

Only for $\xi' = \xi''$ this inner product would be non-zero. Since this is only one point in a given range,

$$\int d\xi' \langle \xi' | \xi'' \rangle = 0 \quad — (7)$$

if the I.P. $\langle \xi' | \xi' \rangle$ is finite. So lets assume that is not. It is also positive as required by properties of I.P. For the normalization (5) to hold

$$\int d\xi' \langle \xi' | \xi'' \rangle$$ has to be

still finite. Such a property defines function $\delta(x)$ such that.

$$\int_{-\infty}^{\infty} \delta(x) = 1 \quad — (8)$$

$$\& \quad \delta(x) = 0 , x \neq 0 \quad — (9)$$

$$\int_{-\infty}^{\infty} \delta(x) f(x) = f(0) \quad — (10)$$

$\delta(x) \rightarrow$ Dirac delta function

$$\therefore \quad \boxed{\langle \xi' | \xi'' \rangle = C\,\delta(\xi' - \xi'')} \quad \text{— (11)}$$

We can normalize $|\xi''\rangle$ easily by defining $k_{\xi'}|\xi'\rangle = |\tilde{\xi}'\rangle$ s.t.

$$\langle \tilde{\xi}' | \tilde{\xi}'' \rangle = k^*_{\xi'} k_{\xi''}\, C\, \delta(\xi' - \xi'')$$

$$\equiv \delta(\xi' - \xi'') \quad \text{— (12)}$$

Since both $|\tilde{\xi}'\rangle$ & $|\xi'\rangle$ are eigenfunction we can drop the tilde and assume normalization!

Thus, $\langle \xi' | \xi'' \rangle = 1_\xi$ — (13)

$$\Rightarrow \int_{-\infty} d\xi' \, |\xi'\rangle\langle\xi'|\xi''\rangle = \int d\xi' \, \delta(\xi' - \xi'') \, |\xi'\rangle$$

$$\Rightarrow \left(\int d\xi' \, |\xi'\rangle\langle\xi'|\right)|\xi''\rangle = |\xi''\rangle \quad \text{— (14)}$$

(from (10))

$$\Rightarrow \quad \boxed{\int d\xi' \, |\xi'\rangle\langle\xi'| = \hat{1}} \quad \text{—(15)}$$

which is the resolution of the identity for a continuous basis set.

We can use (3) & (15) to "represent" arbitrary kets in discrete and continuous basis sets, respectively.

In either case, $\langle\alpha'|\Psi\rangle$ (or $\langle\xi'|\Psi\rangle$) is called the amplitude of $|\Psi\rangle$ (along $|\alpha'\rangle$). In the case of continuous sets this amplitude becomes a continuous function:

$$\Psi(\xi') \equiv \langle\xi'|\Psi\rangle \quad \text{—(16)}$$

This function is called the "wavefunction" corresponding to $|\Psi\rangle$ in $\{|\xi'\rangle\}$ basis.

Functions of observables : If $f(x)$ is a complex function of the real variable x, then we can define the function of an observable $\hat{\alpha}$ as the linear operator which satisfies:

Theorem 4

$$\boxed{f(\hat{\alpha})\,|\alpha'\rangle \equiv f(\alpha')\,|\alpha'\rangle} \quad \text{— (17)}$$

The function can be non-analytic or even non-continuous. We, however, require it to be single-valued.

For an arbitrary state $|\Psi\rangle$,

$$f(\hat{\alpha})\,|\Psi\rangle$$

$$= \sum_{\alpha'} f(\hat{\alpha})\,|\alpha'\rangle\langle\alpha'|\Psi\rangle$$

(using 17)

$$= \sum_{\alpha'} |\alpha'\rangle f(\alpha')\,\langle\alpha'|\Psi\rangle \quad \text{— (18)}$$

$|||^{ly}$ $\langle \alpha' | f(\hat{\alpha})^\dagger \equiv \left(f(\hat{\alpha}) | \alpha' \rangle \right)^{\dagger}$ dual

$= f^*(\alpha') \langle \alpha' |$ — (19)

$\langle \alpha' | f(\hat{\alpha}) | \phi \rangle$

$= \int d\alpha'' \langle \alpha' | f(\hat{\alpha}) | \alpha'' \rangle \langle \alpha'' | \phi \rangle$

$= \int d\alpha'' \, f(\alpha'') \, \delta(\alpha' - \alpha'') \, \phi(\alpha'')$

$= f(\alpha') \phi(\alpha')$ — (13)

(e.s. in continuous set)

<u>Inverse of an operator</u>: When it exists, inverse of $\hat{\alpha}$, denote by $\hat{\alpha}^{-1}$ is such that,

$$\hat{\alpha}^{-1} \hat{\alpha} = \hat{\alpha} \hat{\alpha}^{-1} = \hat{\mathbb{1}}$$ — (14)

N.B.

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1) $(\hat{\alpha}\hat{\beta})^{-1} = \hat{\beta}^{-1}\hat{\alpha}^{-1}$ (prove)

2) $(\hat{\alpha}+\hat{\beta})^{-1} \neq \hat{\alpha}^{-1}+\hat{\beta}^{-1}$

3) $(\hat{\alpha}^{-1})^{-1} = \hat{\alpha}$ (prove)

Compatible observables discussed...
notes follow.