Day 12. Expectation Value & Compatible Observables

26/8/2025

Now let's discuss postulate 4. Let $\hat{\alpha}$ be an observable measured on a state $|\Psi\rangle$.
As we have seen in postulate 3,

$$\hat{\alpha}|\Psi\rangle \longrightarrow \alpha' \underbrace{|\alpha'\rangle}_{\text{eigenstates of } \hat{\alpha}} \quad (16)$$

The arrow in (16) refers to a collapse/jump of $|\Psi\rangle$ to an $|\alpha'\rangle$ state.

Accordingly,
$$|\Psi\rangle = \sum_{\alpha'} c_{\alpha'} |\alpha'\rangle \quad (17)$$

While postulate 3 tells us about the linear dependence of $|\Psi\rangle$ on $\{|\alpha'\rangle\}$, it does not tell how to extract $c_{\alpha'}$ from the measurement.

- P.T.O -

Postulate 4 bridges that gap, at least partially, by telling us that

$|C_{\alpha'}|^2$ = probability that a measurement of $\hat{A}$ will yield the result α'

$$= \frac{\#\text{ of times } \alpha' \text{ resulted}}{\text{total } \# \text{ of independent measurements made.}}$$

(denominator $\to \infty$)

— (18)

This allows us to know the magnitude of the complex number $C_{\alpha'}$. Its phase is undetermined in these measurements. However, thanks to our linear algebra detour, and to the completeness of $\{|\alpha'\rangle\}$, we have that $C_{\alpha'} = \langle\alpha'|\Psi\rangle$ — (19)

Thus, postulate 4 gives a physical meaning to the inner product between 2 states.

$\langle \alpha' | \Psi \rangle \longrightarrow$ probability amplitude of a measurement of $\hat{\alpha}$ on $|\Psi\rangle$ yielding the result α'.

From postulate 3 b), this is also the amplitude associated with finding the system, originally prepared in $|\Psi\rangle$, in the state $|\alpha'\rangle$.

More generally we can say that the inner product $\langle \Phi | \Psi \rangle$ yields the probability amplitude that a system, originally prepared in $|\Psi\rangle$, would be found in $|\Phi\rangle$ following the appropriate measurement.

Since, states can be time dependent, so can the amplitudes. e.g.

$$C_{\alpha'}(t) = \langle \alpha' | \Psi(t) \rangle \quad \text{— (20)}$$

This postulate (also called Born's rule) allows us to compute the average value of an observable $\hat{\alpha}$ in a state $|\Psi\rangle$ as an expectation value.

$$\langle \hat{\alpha} \rangle \equiv \sum_{\alpha'} P_{\alpha'} \, \alpha' \quad \text{— (21)}$$

$P_{\alpha'}$ → probability of α' result

$\therefore$

$$\boxed{\langle \hat{\alpha} \rangle = \sum_{\alpha'} |\langle \alpha' | \Psi \rangle|^2 \, \alpha'} \quad \text{— (22)}$$

(Using (18) & (19))

Eq. (22) presumes that $|\alpha'\rangle$ are normalized.

Explanation. Average of measurements of a random variable x:

$$\bar{x} \equiv \langle x \rangle = \frac{1}{N} \sum_{i=1}^{N} x_i$$

N → # of measurements

x_i → value in i^{th} measurement

$$= \sum_{x \in \text{all possible results}} \frac{M(x)}{N} \, x \xrightarrow{N \to \infty} \sum_{x} P(x) \, x \quad \text{— (23)}$$

$M(x)$ → # of times value x measured

Now, consider

$$\langle \Psi | \hat{\alpha} | \Psi \rangle$$

$$\hat{1} = \sum_{\alpha'} |\alpha'\rangle\langle\alpha'| \qquad \hat{1} = \sum_{\alpha''} |\alpha''\rangle\langle\alpha''|$$

$$= \sum_{\alpha'} \sum_{\alpha''} \langle \Psi | \alpha' \rangle \langle \alpha' | \hat{\alpha} | \alpha'' \rangle \langle \alpha'' | \Psi \rangle$$

$$= \sum_{\alpha'} \sum_{\alpha''} \alpha' \, \delta_{\alpha'\alpha''} \, |\langle \alpha' | \Psi \rangle|^2$$

$$= \sum_{\alpha'} \alpha' \, |\langle \alpha' | \Psi \rangle|^2 \quad \text{— (24)}$$

(22) & (24) are exactly the same

$\therefore$ we have that the average value of an observable $\hat{\alpha}$ in a state $|\Psi\rangle$ is given by the expectation value

$$\boxed{\bar{\alpha} = \langle \Psi | \hat{\alpha} | \Psi \rangle} \quad \text{— (25)}$$

If $|\Psi\rangle$ is not normalized then

$$\boxed{\bar{\alpha} = \frac{\langle\Psi|\hat{\alpha}|\Psi\rangle}{\langle\Psi|\Psi\rangle}} \qquad (26)$$

For continuous spectrum of eigenvalues $\{\xi'\}$ the corresponding expressions are:

Probability amplitude:

$$\Psi(\xi') \equiv \langle\xi'|\Psi\rangle \qquad (27)$$

$$\bar{\alpha} = \int d\xi' \, |\langle\xi'|\Psi\rangle|^2 \, \xi'$$

$$= \int d\xi' \, |\Psi(\xi')|^2 \, \xi' \qquad (28)$$

As a special case of continuous eigenvalues consider the position operator $\hat{\vec{r}}$.

Average position of a particle in a state $|\Psi\rangle$ (at a particular time) is $\langle\hat{\vec{r}}\rangle = \langle\Psi|\hat{\vec{r}}|\Psi\rangle$

$$= \int_{-\infty}^{\infty}dx\int_{-\infty}^{\infty}dy\int_{-\infty}^{\infty}dz\ |\Psi(\vec{r},t)|^2\,\vec{r}$$

$$= \int_{\text{all space}} d^3r\ |\Psi(\vec{r},t)|^2\,\vec{r} \quad \text{— (29)}$$

Similarly, if an observable depends only on the particle position, i.e. $\hat{\alpha} \equiv \alpha(\hat{\vec{r}})$ — (30)

then by Theorem 4 (17 in prev. lecture)

$$\hat{\alpha}\,|\vec{r}\,'\rangle = \alpha(\vec{r}\,')\,|\vec{r}\,'\rangle \quad \text{— (30)}$$

$$\Rightarrow \quad \langle\vec{r}\,''|\hat{\alpha}|\vec{r}\,'\rangle = \alpha(\vec{r}\,')\,\langle\vec{r}\,''|\vec{r}\,'\rangle$$

$\hookrightarrow$ number

$$= \alpha(\vec{r}\,')\,\delta(\vec{r}\,''-\vec{r}\,') \quad \text{— (31)}.$$

$$\therefore \quad \langle \hat{\alpha} \rangle = \langle \Psi | \overset{\hat{1}}{\swarrow} \hat{\alpha} \overset{\hat{1}}{\swarrow} | \Psi \rangle$$

$$= \int d^3r' \int d^3r'' \, \langle \Psi | \vec{r}\,'' \rangle \langle \vec{r}\,'' | \hat{\alpha} | \vec{r}\,' \rangle \langle \vec{r}\,' | \Psi \rangle$$

$$= \int d^3r' \int d^3r'' \, \Psi^*(\vec{r}\,'') \, \alpha(\vec{r}\,') \, \delta(\vec{r}\,' - \vec{r}\,'') \cdot \Psi(\vec{r}\,')$$

$$\boxed{= \int d^3r' \, \Psi^*(\vec{r}\,') \, \alpha(\vec{r}\,') \, \Psi(\vec{r})} \quad (32)$$

We will see more about objects such as $\langle \vec{r}\,' | \hat{\alpha} | \vec{r}\,'' \rangle$ later.

Compatible Observables :

Two observables that commute are said to be mutually compatible. Compatibility refers to the fact that for such observables, measurements of one does not preclude the precise measurement of the other.

$$\underbrace{\bullet \xrightarrow{\hat{\alpha}} |\alpha'\rangle}_{\text{State preparation}} \xrightarrow{\hat{\beta}} |\beta'\rangle \xrightarrow{\hat{\alpha}} |\alpha'\rangle \xrightarrow{\hat{\beta}} |\beta'\rangle \longrightarrow \cdots$$

This also means that the order in which these measurements are made does not affect the eigenstate into which the system jumps.

Mathematically,

$$\boxed{[\hat{\alpha}, \hat{\beta}] = 0} \qquad \text{--- (33)}$$