1/9/2025 Day 13. Compatible & incompatible observables and the Uncertainty principle

Theorem 5: If 2 observables commute then there exists a complete set of simultaneous eigenstates of the two.

$\Rightarrow \exists \ \{|\alpha', \beta'\rangle\}$ with $\hat{\alpha}\,|\alpha', \beta'\rangle = \alpha' \,|\alpha', \beta'\rangle$

$\hat{\beta}\,|\alpha', \beta'\rangle = \beta' \,|\alpha', \beta'\rangle$

such that any state $|\Psi \in \mathcal{H}$ can be written as:

$$\boxed{|\Psi\rangle = \sum_{\alpha'} \sum_{\beta'} |\alpha', \beta'\rangle\langle \alpha', \beta' | \Psi\rangle} \quad — (1)$$

Proof: Consider 2 observables $\hat{\alpha}, \hat{\beta}$ such that $[\hat{\alpha}, \hat{\beta}] = 0$ — (2)

Let
$$\left.\begin{aligned} \hat{\alpha}\,|\alpha'\rangle &= \alpha' \,|\alpha'\rangle \\ \hat{\beta}\,|\hat{\beta}\rangle &= \beta' \,|\beta'\rangle \end{aligned}\right\} (3)$$

$\{|\alpha'\rangle\}, \{|\beta'\rangle \rightarrow$ o/n.

Case 1: α' are all non-degenerate.

$$② \Rightarrow \langle\alpha''|\,[\hat\alpha,\hat\beta]\,|\alpha'\rangle = 0$$

or $\langle\alpha''|\hat\alpha\hat\beta|\alpha'\rangle - \langle\alpha''|\hat\beta\hat\alpha|\alpha'\rangle = 0$

i.e. $(\alpha''-\alpha')\langle\alpha''|\hat\beta|\alpha'\rangle = 0 \quad — ④$

If $\alpha''\neq\alpha'$ then $\langle\alpha''|\hat\beta|\alpha'\rangle = 0 \quad ⑤$

This result is itself stated as theorem.

Theorem: If $[\hat\alpha,\hat\beta]=0$ and $\hat\alpha$ has non-degenerate eigenvalues, then $\hat\beta$ is diagonal in the basis of $\{|\alpha'\rangle\}$ states. i.e. $\langle\alpha''|\hat\beta|\alpha'\rangle = 0$, $\alpha''\neq\alpha$.

Now, $$\hat\beta|\alpha'\rangle = \sum_{\alpha''}|\alpha''\rangle\langle\alpha''|\hat\beta|\alpha'\rangle$$

$\hat{\mathbb{1}}$

$$= \langle\alpha'|\hat\beta|\alpha'\rangle\;|\alpha'\rangle \text{ (using ⑤)}$$

$$= \beta'(\alpha')\;|\alpha'\rangle \quad — ⑥$$

(6) is just the eigenvalue equation for $\hat{\beta}$. Thus, $\hat{\alpha}$ & $\hat{\beta}$ have simultaneous eigenstates, which we now label as $\{|\alpha', \beta'\rangle\}$.

Note that while we have assumed $\hat{\alpha}$ to be non-degenerate, there is no such restriction on $\hat{\beta}$. Therefore, in principle, there could be more than one α' for which β' is the same. This justifies the double-label of the states. Since they are eigenstates of an observable we have:

$$\boxed{\sum_{\alpha', \beta'} |\alpha' \beta'\rangle\langle\alpha' \beta'| = \hat{\mathbb{1}}} \quad \text{— (7)}$$

Hence proved.

Case 2: α' is d-fold degenerate.
Let the corresponding states be $\{|\alpha'_j\rangle\}_{j=1,d}$. O/N — (7.5)

For states $|\alpha''\rangle$, not in this degenerate set eqn. (5) still holds. But for states within the set, in general, $\langle\alpha_j'|\hat{\beta}|\alpha_i'\rangle \neq 0$. — (8)

However, given a set of linearly independent states $\{|\alpha_j'\rangle\}_{j=1,d}$, it is possible to come up with another set $\{|\tilde{\alpha}_j'\rangle\}_{j=1,d}$ such that

$$\hat{\alpha}\,|\tilde{\alpha}_j'\rangle = \alpha'\,|\tilde{\alpha}_j'\rangle \quad -(9)$$

$$\& \quad \langle\tilde{\alpha}_j'|\hat{\beta}|\tilde{\alpha}_i'\rangle = \delta_{ij} \quad -(10)$$

This can be done by insisting there exist $\{C_{ik}\}$ such that

$$|\tilde{\alpha}_i'\rangle = \sum_{k=1}^{d} C_{ik}\,|\alpha_k'\rangle \quad -(11)$$

and (10) is satisfied.

Note that, if we require, $\{|\tilde{\alpha}_j'\rangle\}$ to be o/n. then it means:

$$\langle\tilde{\alpha}_i'|\tilde{\alpha}_j'\rangle = \delta_{ij} \qquad \text{— (12)}$$

$$\sum_{k=1}^{d}\sum_{l=1}^{d} C_{ik}^* \underbrace{\langle\alpha_k'|\alpha_l'\rangle}_{\delta_{kl}} C_{jl} = \delta_{ij} \qquad \text{— (13)}$$

$$\therefore \sum_{k=1}^{d} C_{ik}^* C_{jk} = \delta_{ij} \qquad \text{— (14)}$$

$\forall\, i,k = 1, d.$

Note that the connection between $|\alpha'\rangle$ & $|\tilde{\alpha}'\rangle$ basis can be made formally through an operator $\hat{U}$, s.t

$$|\tilde{\alpha}_i'\rangle = \hat{U}\,|\alpha_i'\rangle \qquad \text{— (15)}$$

(12) $\Rightarrow$

$$\langle\tilde{\alpha}_i'|\tilde{\alpha}_j'\rangle = \langle\alpha_i'|\hat{U}^\dagger\hat{U}|\alpha_j'\rangle \qquad \text{— (16)}$$

Since this a requirement for any (i,j) we have that

$$\hat{U}^\dagger \hat{U} = \hat{\mathbb{1}} \quad \text{— (17)}$$

$$\Rightarrow \hat{U}^{-1} = \hat{U}^\dagger \quad \text{— (18)}$$

(15) $\Rightarrow$
$$|\alpha_i'\rangle = \hat{U}^{-1}|\tilde{\alpha}_i'\rangle = \hat{U}^\dagger|\tilde{\alpha}_i'\rangle \quad \text{— (19)}$$

$\therefore \langle\alpha_i'|\alpha_j'\rangle = \delta_{ij} \Rightarrow$

$$\langle\tilde{\alpha}_i'|\hat{U}\hat{U}^\dagger|\tilde{\alpha}_j'\rangle = \delta_{ij} \quad \text{— (20)}$$

$$\Rightarrow \hat{U}\hat{U}^\dagger = \hat{\mathbb{1}} \quad \text{— (21)}$$

$$\therefore \boxed{\hat{U}\hat{U}^\dagger = \hat{U}^\dagger\hat{U} = \hat{\mathbb{1}}} \quad \text{— (22)}$$

Such an operator is called a unitary operator.

Unitary operators are often used for changing from one O/N to another O/N basis.

So eq. (10) can be written as

$$\langle \alpha_i' | \hat{U}^\dagger \hat{\beta} \hat{U} | \alpha_j' \rangle = \delta_{ij} \quad \text{--- (23)}$$

Or

$$\boxed{\hat{U}^\dagger \hat{\beta} \hat{U} = \hat{1}} \quad \text{--- (24)}$$

A unitary operator which achieves (24) while acting on $\hat{\beta}$ is said to diagonalize $\hat{\beta}$.

We will see more of this shortly. The outcome, however, is that even when the α' is non-degenerate we can show that, by an appropriate "unitary transformation", we can get a set of L.I. states corresponding to the eigenvalue α', where the

$$\langle \tilde{\alpha}_i' | \hat{\beta} | \tilde{\alpha}_j' \rangle = 0 \quad \text{if } i \neq j \quad \text{--- (25)}$$

The diagonal values now correspond to eigenvalues of $\hat{\beta}$ when $\hat{\alpha}$ yields α'. These states are then labeled as

$$|\tilde{\alpha}_1', \beta_1\rangle, |\tilde{\alpha}_2', \beta_2\rangle \ldots, |\tilde{\alpha}_d', \beta_d\rangle$$

where $\beta_1, \cdots \beta_d$ are not necessarily equal.

This says that while an $\hat{\alpha}$ measurement could not distinguish between these states (same α' eigenvalue), a $\hat{\beta}$ measurement can. Thus, we have "resolved" the degeneracy.

The idea could be extended to other compatible observable $\hat{\gamma}, \hat{\delta}$, etc., such that

$$[\hat{\alpha}, \hat{\beta}] = [\hat{\beta}, \hat{\gamma}] = [\hat{\gamma}, \hat{\delta}]$$
$$= [\hat{\alpha}, \hat{\gamma}] = \cdots = 0 \quad \text{ie}$$

they all mutually commute and are hence mutually compatible.

Then the base states of the system can be completely resolved $\{|\alpha', \beta', \gamma', \delta', \ldots\rangle\}$

if this set of mutually compatible observables is complete.

Incompatible Observables and the uncertainty principle

Theorem 6: If 2 observables are incompatible then they do not share a complete set of simultaneous eigenstates.

(easy to prove)

$\Rightarrow$ Measurements will not yield both with arbitrary precision simultaneously.

Let's quantify this inference.

Let's define the operator

$$\hat{\Delta\alpha} \equiv \hat{\alpha} - \langle\hat{\alpha}\rangle \quad \text{--- (26)}$$

$\langle\hat{\alpha}\rangle \rightarrow$ expectation value in an arbitrary physical state $|\Psi\rangle$.

Then, $\langle(\Delta\hat{\alpha})^2\rangle$

gives us the spread of α' values that measurements would yield

$$\langle(\Delta\hat{\alpha})^2\rangle = \langle(\hat{\alpha} - \langle\alpha\rangle)^2\rangle$$

$$= \langle\hat{\alpha}^2 - 2\langle\hat{\alpha}\rangle^2 + \langle\hat{\alpha}\rangle^2\rangle$$

$$= \langle\hat{\alpha}^2\rangle - \langle\hat{\alpha}\rangle^2 \quad \text{--- (27)}$$

where we have used the formula

$$\langle\hat{\alpha}\rangle = \langle\Psi|\hat{\alpha}|\Psi\rangle \quad \text{--- (28)}$$

$|\Psi\rangle \rightarrow$ normalized

By postulate 4, (27) is just the variance of the distribution of α'.

Theorem 9: Given 2 observable $\hat{\alpha}$ & $\hat{\beta}$, their variances in any physical state satisfies

$$\boxed{\langle(\Delta\hat{\alpha})^2\rangle\langle(\Delta\hat{\beta})^2\rangle \geq \frac{1}{4}\left|\langle[\hat{\alpha},\hat{\beta}]\rangle\right|^2} \quad -(28)$$

To prove this we will use three lemmas from vector spaces.

I. Schwarz inequality of vector spaces (prove as H.W).

$$\boxed{\langle u|u\rangle\langle v|v\rangle \geq \left|\langle u|v\rangle\right|^2} \quad -(29)$$

where $|u\rangle, |v\rangle \in \mathcal{H}$.

II. Expectation value of a Hermitian operator is purely real.

III. Expectation value of an anti-Hermitian operator is purely imaginary

Now we prove the theorem by first setting

$$\left.\begin{aligned} |U\rangle &= \Delta\hat{\alpha}\,|\Psi\rangle \\ |V\rangle &= \Delta\hat{\beta}\,|\Psi\rangle \end{aligned}\right\} \quad (30)$$

in (29)

This gives us:

$$\langle(\Delta\hat{\alpha})^2\rangle\langle(\Delta\hat{\beta})^2\rangle \geq \left|\langle\Delta\hat{\alpha}\,\Delta\hat{\beta}\rangle\right|^2 \quad -(31)$$

where we have used the fact that $\Delta\hat{\alpha}$, $\Delta\hat{\beta}$ are Hermitian and, hence,

$$\Delta\hat{\alpha}^{\dagger} = \Delta\hat{\alpha} \;\&\; \Delta\hat{\beta}^{\dagger} = \Delta\hat{\beta} \quad -(32)$$

Now, $$\Delta\hat{\alpha}\,\Delta\hat{\beta} = \frac{1}{2}\left[\,[\Delta\hat{\alpha},\Delta\hat{\beta}] + \{\Delta\hat{\alpha},\Delta\hat{\beta}\}\,\right] \quad -(33)$$

where $$\{\hat{\alpha},\hat{\beta}\} \equiv \hat{\alpha}\hat{\beta} + \hat{\beta}\hat{\alpha} \quad -(34)$$

(anti-commutator)

Furthermore,

$$[\Delta\hat{\alpha}, \Delta\hat{\beta}] = [\hat{\alpha} - \langle\hat{\alpha}\rangle, \hat{\beta} - \langle\hat{\beta}\rangle]$$

$$= [\hat{\alpha}, \hat{\beta}] \quad — (35)$$

We also note that the commutator is an anti-Hermitian operator.

$$[\hat{\alpha}, \hat{\beta}]^\dagger = -[\hat{\alpha}, \hat{\beta}] \quad — (36)$$

by its definition.

But the anti-commutator is Hermitian.

$$\{\hat{\alpha}, \hat{\beta}\}^\dagger = \{\hat{\alpha}, \hat{\beta}\} \quad — (37)$$

The RHS of (37) becomes:

$$|\langle\Delta\hat{\alpha}\Delta\hat{\beta}\rangle|^2 = \frac{1}{4}\left|\underbrace{\langle[\hat{\alpha}, \hat{\beta}]\rangle}_{\text{purely imaginary (Lemma III)}} + \underbrace{\langle\{\Delta\hat{\alpha}, \Delta\hat{\beta}\}\rangle}_{\text{purely real (Lemma II)}}\right|^2 \quad — (38)$$

$$= \frac{1}{4}\left|\langle[\hat{\alpha},\hat{\beta}]\rangle\right|^2 + \frac{1}{4}\left|\langle\{\Delta\hat{\alpha},\Delta\hat{\beta}\}\rangle\right|^2 \quad \text{— (39)}$$

(since $|a+ib|^2 = |a|^2+|b|^2$)

Second term in (39) is positive so dropping it does not change the inequality in (31) but only makes it stronger.

Therefore, we have

$$\boxed{\langle(\Delta\hat{\alpha})^2\rangle\langle(\Delta\hat{\beta})^2\rangle \geq \frac{1}{4}\left|\langle[\hat{\alpha},\hat{\beta}]\rangle\right|^2} \quad \text{— (40)}$$

Or taking square root on both sides:

$$\boxed{\Delta\alpha\,\Delta\beta \geq \frac{1}{2}\left|\langle[\hat{\alpha},\hat{\beta}]\rangle\right|} \quad \text{— (41)}$$

where $$\left.\begin{aligned} \Delta \alpha &\equiv \sqrt{\langle (\Delta \hat{\alpha})^2 \rangle} \\ \Delta \beta &\equiv \sqrt{\langle (\Delta \hat{\beta})^2 \rangle} \end{aligned}\right\} \quad (4.2)$$

are variances of the respective distribution also termed as uncertainties in the respective variables

(41) is the generalised uncertainty principle.

If we know the commutator of 2 observables then we can predict the uncertainty relation for them.

For all mechanical observables, it turns out that, it is enough to know just one fundamental commutator.

Postulate 5: The position and momentum operators of any (quantum) particle satisfy the following commutator relations.

$$[r_\alpha , p_\beta] = i\hbar\, \delta_{\alpha\beta} \qquad \text{— (43)}$$

where α, β run over the Cartesian components x, y, z.

Using this postulate and the G.U.P. (41) we can write.

$$\Delta r_\alpha \Delta p_\alpha \geq \frac{\hbar}{2}\, \delta_{\alpha\beta} \qquad \text{— (44)}$$

or

$$\boxed{\begin{aligned} \Delta x \Delta p_x &\geq \frac{\hbar}{2} \\ \Delta y \Delta p_y &\geq \frac{\hbar}{2} \\ \Delta z \Delta p_z &\geq \frac{\hbar}{2} \end{aligned}} \qquad \text{— (45)}$$