10/9/2025 Day 19. Solutions for time-independent problems

Time-independent Hamiltonian:

Let us specialize to the case where the interactions are not dep. on time. The Hamiltonian is also time indep. Since the Hamiltonian is Hermitian, its eigenstates would provide a convenient representation for the state.

$$\hat{\mathcal{H}}|E\rangle = E|E\rangle \quad \text{— (10)}$$

$$\langle E|E'\rangle = \delta_{EE'} \quad \text{— (10.5)}$$

$$\therefore \quad E = \langle E|\hat{\mathcal{H}}|E\rangle \quad \text{— (11)} \quad (|E\rangle \text{ norm.})$$

E corresponds to energies of the system when in the state $|E\rangle$.

Now $$|E\rangle\langle E| = \hat{\mathbb{1}} \quad \text{— (12)}$$

So we can write $|\Psi(t)\rangle = \sum_E a_E(t)|E\rangle$

where $a_E(t) = \langle E|\Psi(t)\rangle$

(13)

Using (13) in (7) we get:

$$\sum_{E'} i\hbar \frac{\partial a_{E'}(t)}{\partial t} |E'\rangle = \sum_{E'} \hat{H} |E'\rangle a_{E'}(t)$$

$$= \sum_{E} E' |E'\rangle a_{E'}(t) \quad \text{— (14)}$$

Taking innerproduct with $\langle E|$ this yields:

$$i\hbar \frac{d}{dt} a_E(t) = E\, a_E(t) \quad \text{— (15)}$$

$$\Rightarrow \quad a_E(t) = a_E(0) \exp\left(-i\frac{E}{\hbar}t\right) \quad \text{— (16)}$$

(1st order linear homogeneous differential equation with initial condition
$$a_E(t=0) = a_E(0)$$
)

$$\therefore \quad \boxed{|\Psi(t)\rangle = \sum_{E} |E\rangle a_E(0)\, e^{-i\frac{Et}{\hbar}}} \quad \text{— (17)}$$

Therefore, the standard protocol to solve the TDSE for any system is as follows:

1. Write down the Hamiltonian for the system $\hat{H}$. (time-indep.)

2. Set up and solve the eigenvalue equation for the Hamiltonian (the time-independent SE or TISE).

$$\hat{H}\,|E\rangle = E\,|E\rangle \quad -(18)$$

3. Determine the initial value of the energy coefficients given $|\Psi(0)\rangle$

$$a_E(0) = \langle E|\Psi(0)\rangle \quad -(19)$$

4. Construct the time-dependent state as

$$|\Psi(t)\rangle = \sum_E{}' \, a_E(0)\, e^{-i\frac{Et}{\hbar}}\,|E\rangle \quad -(20)$$

We will, for the most of this course, follow this procedure.

(20) says that any system is, at any instant, in a superposition of its energy eigenstates. This brings special importance to the TISE and, hence, we will be occupied with solving this equation for many model systems.

However, (18) is still an abstract equation. Let us choose a representation to solve it. The conventional choice is the position representation.

Let us, for the present, focus on a 1-particle system with a particle of mass m moving along the x-axis.

Then, $$\hat{H} = \frac{\hat{p}_x^2}{2m} + V(\hat{x}) \quad — (21)$$

$$\hat{H}|E\rangle = E|E\rangle$$

$$\Rightarrow \quad \langle x|\hat{H}|E\rangle = E\underbrace{\langle x|E\rangle}_{= \psi_E(x)} \quad — (22)$$

$\hookrightarrow = \psi_E(x)$ — (23)

LHS of (22)

$$= \frac{1}{2m}\langle x|\hat{p}_x^2|E\rangle + \langle x|V(\hat{x})|E\rangle$$

$$= -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\langle x|E\rangle + V(x)\langle x|E\rangle$$

$$= \left\{-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)\right\}\psi(x) \quad — (24)$$

RHS of (58) $= E\,\psi_E(x)$ — (25)

$\therefore$ we get

$$\boxed{\left\{-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)\right\}\psi_E(x) = E\,\psi_E(x)} \quad — (26)$$