7/10/2025 Day 24 Continuity Equation

In the context of QM, a similar conservation property holds for the probability, for instance, of finding a particle in a volume d^3r around $\vec{r}$ at any time. This is ensured by normalization of the state.

$$\langle \Psi(t) | \Psi(t) \rangle = 1 \qquad — (24)$$

$$\Rightarrow 1 = \int d^3r \, \langle \Psi(t) | \vec{r} \rangle \langle \vec{r} | \Psi(t) \rangle$$

$$= \int d^3r \, \Psi^*(\vec{r},t) \Psi(\vec{r},t)$$

$$\equiv \int d^3r \, P(\vec{r},t) \qquad — (25)$$

probability density. (like charge density)

So can we derive an analogous continuity equation in QM?

Yes.

Consider the TDSE in position representation. (for 1-particle in 3-d)

$$i\hbar \frac{\partial}{\partial t} \Psi(\vec{r},t) = -\frac{\hbar^2}{2m} \nabla^2 \Psi(\vec{r},t) + V(\vec{r},t)\Psi(\vec{r},t) \quad \text{—} (26)$$

Taking complex conjugate on both sides we get.

$$-i\hbar \frac{\partial}{\partial t} \Psi^*(\vec{r},t) = -\frac{\hbar^2}{2m} \nabla^2 \Psi^*(\vec{r},t) + V(\vec{r},t)\Psi^*(\vec{r},t) \quad \text{—} (27)$$

Here, V is assumed to be real as is required for Hermiticity of $\hat{H}$.

$\Psi^* \times (26) - \Psi \times (27) \Rightarrow$

$$i\hbar \left[\Psi^* \frac{\partial \Psi}{\partial t} + \Psi \frac{\partial \Psi^*}{\partial t} \right] = -\frac{\hbar^2}{2m} \left[\Psi^* \nabla^2 \Psi - \Psi \nabla^2 \Psi^* \right] \quad \text{—} (28)$$

$$\Rightarrow \quad i\hbar \frac{\partial P(\vec{r},t)}{\partial t}$$

$$= \frac{-\hbar^2}{2m}\left[\Psi^* \nabla^2 \Psi + (\vec{\nabla}\Psi^*)\cdot(\vec{\nabla}\Psi) - (\vec{\nabla}\Psi)\cdot(\vec{\nabla}\Psi)^* - \Psi \nabla^2 \Psi^* \right]$$

$$= \frac{-\hbar^2}{2m}\left[\vec{\nabla}\cdot\left(\Psi^* \vec{\nabla}\Psi - \Psi \vec{\nabla}\Psi^* \right) \right] \quad \text{—— (29)}$$

where we have used the identity.

$$\left.\begin{aligned} \vec{\nabla}\cdot\left(f(\vec{r})\vec{A}(\vec{r}) \right) &= \vec{\nabla} f(\vec{r})\cdot\vec{A}(\vec{r}) \\ &\quad + f(\vec{r})\,\vec{\nabla}\cdot\vec{A}(\vec{r}) \\ \& \quad \vec{\nabla}\cdot\vec{\nabla} f &= \nabla^2 f \end{aligned}\right\} \quad (30)$$

(29) $\Rightarrow$ $$\boxed{\frac{\partial P(\vec{r},t)}{\partial t} = -\vec{\nabla}\cdot\vec{j}(\vec{r},t)} \quad (31)$$

where
$$\boxed{\vec{j}(\vec{r},t) \equiv \frac{\hbar}{2mi}\left[\Psi^*(\vec{r},t)\vec{\nabla}\Psi(\vec{r},t) - \Psi(\vec{r},t)\vec{\nabla}\Psi^*(\vec{r},t)\right]} \quad (32)$$

is the probability current density.

Since $z - z^* = 2i\,\mathrm{Im}(z)$ (33)

we can also write (32) as

$$\boxed{\vec{j}(\vec{r},t) = \mathrm{Im}\left[\frac{\hbar}{m}\Psi^*(\vec{r},t)\vec{\nabla}\Psi(\vec{r},t)\right]} \quad (34)$$

Since total probability is always conserved,

$$\lim_{V\to\infty} \frac{d}{dt} \int_V d^3r \, P(\vec{r},t) = -\int_V d^3r \, \vec{\nabla}\cdot\vec{j}(\vec{r},t)$$

$$= -\int_{S(V)} \vec{j}(\vec{r},t)\cdot d\vec{S}$$

$$= 0 \qquad \text{--- (35)}$$

$\Rightarrow$ The net probability out-flux at the boundaries (infinities) vanishes and, even, locally,

$$\lim_{|\vec{r}|\to\infty} \vec{j}(\vec{r},t) \to 0 \qquad \text{--- (36)}$$

Examples.

(1) $$\psi_{\vec{p}}(\vec{r},t) = \left(\frac{1}{\sqrt{2\pi\hbar}}\right)^3 \exp\left(i\left(\frac{\vec{p}\cdot\vec{r}}{\hbar} - \omega_{\vec{p}}\, t\right)\right)$$

where $\omega_p = \dfrac{p^2}{2\hbar m}$ — (37)

$$\vec{\nabla}\psi_{\vec{p}}(\vec{r},t) = \left(\frac{i\vec{p}}{\hbar}\right)\psi_{\vec{p}}(\vec{r},t) \quad \text{— (38)}$$

$$\Rightarrow \psi^*_{\vec{p}}(\vec{r},t)\,\vec{\nabla}\psi_{\vec{p}}(\vec{r},t) = \frac{i\vec{p}}{\hbar}\left(\frac{1}{2\pi\hbar}\right)^3$$

$$\Rightarrow \vec{J}(\vec{r},t) = \mathrm{Im}\left\{\frac{\hbar}{m}\psi^*_{\vec{p}}(\vec{r},t)\,\vec{\nabla}\psi_{\vec{p}}(\vec{r},t)\right\}$$

$$= \frac{\vec{p}}{m}\cdot\frac{1}{(2\pi\hbar)^3} \quad \text{— (39)}$$

$\Rightarrow \vec{\nabla}\cdot\vec{J} = 0 \quad \Longleftrightarrow \quad \rho(\vec{r},t) = 1/(2\pi\hbar)^3$

$$\Rightarrow \frac{\partial\rho(\vec{r},t)}{\partial t} = 0$$

Note that (39) also means.

$$\vec{j}(\vec{r},t) = \rho(\vec{r},t)\left(\frac{\vec{p}}{m}\right) = \rho(\vec{r},t)\,\vec{v} \quad -(40)$$

which agrees with the classical notion of current density.

(2) $$\psi(\vec{r},t) = \phi(\vec{r})\, e^{-iE/\hbar\, t} \quad -(41)$$

$$\vec{\nabla}\psi(\vec{r},t) = \left(\vec{\nabla}\phi(\vec{r})\right) e^{-iE/\hbar\, t} \quad -(42)$$

$$\Rightarrow \psi^* \vec{\nabla}\psi = \phi^*(\vec{r})\,\vec{\nabla}\phi(\vec{r}) \quad -(43)$$

If $\phi(\vec{r}) \in \mathbb{R}$, then $\psi^* \vec{\nabla}\psi \in \mathbb{R}$.

$$\Rightarrow \boxed{\vec{j}(\vec{r},t) = 0} \quad -(44)$$

Thus, the current density associated with real wavefunctions or wavefunction with only separable complex time parts is zero.

③

$$\psi(x,t) = A(t)\, e^{ipx/\hbar} + B(t)\, e^{-ipx/\hbar}$$

$$\psi^* \nabla \psi = \left(A^* e^{-ipx/\hbar} + B^* e^{ipx/\hbar}\right)$$

$$\frac{ip}{\hbar}\left(A\, e^{ipx/\hbar} - B\, e^{-ipx/\hbar}\right)$$

$$= i\left(|A|^2 - |B|^2\right)\frac{p}{\hbar}$$

$$+ \, ip/\hbar \left[AB^* e^{i2px/\hbar} - A^* B e^{-i2px/\hbar}\right]$$

$$= i\left(|A|^2 - |B|^2\right)p/\hbar$$

$$- 2p/\hbar \, \mathrm{Im}\left\{AB^* e^{i2px/\hbar}\right\} \quad \text{— (45)}$$

$$\Rightarrow \frac{\hbar}{m}\mathrm{Im}\left\{\psi^* \nabla \psi\right\} = \left(|A|^2 - |B|^2\right)\frac{p}{m}$$

$$= j(x,t), \quad \text{— (46)}$$