14/10/2025 Lecture 25. QMSHO – The Ladder Operator Approach

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2} m\omega^2 \hat{x}^2 \quad \text{— (1)}$$

Let $$\hat{\xi} \equiv \hat{x} \Big/ \sqrt{\frac{\hbar}{m\omega}} \quad \text{— (2)}$$

& $$\hat{\pi} \equiv \hat{p} \Big/ \sqrt{m\omega\hbar} \quad \text{— (3)}$$

$$[\hat{\xi}, \hat{\pi}] = \frac{1}{\hbar}[\hat{x}, \hat{p}] = i \quad \text{— (4)}$$

Let $$\hat{a}_{\pm} = \hat{\xi} \mp i\hat{\pi} \quad \text{— (5)}$$

$$\Rightarrow \quad \hat{a}_+^{\dagger} = \hat{a}_- \quad \text{— (6)}$$

$$\Rightarrow \quad [\hat{a}_+, \hat{a}_-] = [\hat{\xi} - i\hat{\pi}, \hat{\xi} + i\hat{\pi}]$$

$$= -i[\hat{\pi}, \hat{\xi}] + i[\hat{\xi}, \hat{\pi}]$$

$$= -1$$

or. $$[\hat{a}_-, \hat{a}_+] = 1 \quad \text{— (7)}$$

$$\Rightarrow \quad \hat{H} = \left[\hat{a}_+\hat{a}_- + \frac{1}{2}\right]\hbar\omega \quad - (8)$$

Let $\quad \hat{N} = \hat{a}_+\hat{a}_- \quad - (9)$

$$[\hat{a}_-, \hat{N}] = [\hat{a}_-, \hat{a}_+]\hat{a}_- = \hat{a}_- \quad - (10)$$

lllly $\quad [\hat{a}_+, \hat{N}] = -\hat{a}_+ \quad - (11)$

$$\therefore \quad [\hat{a}_\mp, \hat{H}] = \pm\hbar\omega\,\hat{a}_\mp \quad - (12)$$

Now,

$$\hat{H}|E\rangle = E|E\rangle \quad - (13)$$

$$\Rightarrow \quad \hat{a}_+\hat{H}|E\rangle = E\,\hat{a}_+|E\rangle \quad - (14)$$

using (12) this becomes

$$-\hbar\omega(\hat{a}_+|E\rangle) + \hat{H}(\hat{a}_+|E\rangle) = E(\hat{a}_+|E\rangle)$$

or $\quad \hat{H}(\hat{a}_+|E\rangle) = (E+\hbar\omega)(\hat{a}_+|E\rangle) \quad - (15)$

$\Rightarrow$ $\hat{a}_+|E\rangle$ is an eigenstate of $\hat{H}$ with the eigenvalue $E+\hbar\omega$.

This is easily extended to show that $\hat{a}_+^n|E\rangle$ is a eigenstate of $\hat{H}$ with eigenvalue $(E + n\hbar\omega)$.

IIIly we can show that

$$\hat{H}\left(\hat{a}_-|E\rangle\right) = (E - \hbar\omega)\left(\hat{a}_-|E\rangle\right) \quad \text{— (16)}$$

$\Rightarrow$ $\hat{a}_-|E\rangle$ is an eigenstate of $\hat{H}$ with eigenvalue $E - \hbar\omega$.

And, $\hat{a}_-^n|E\rangle$ yields the eigenvalue $(E - n\hbar\omega)$.

Thus, $\hat{a}_+$ and $\hat{a}_-$ are called raising and lowering operators as they step up or down the energy in units of $\hbar\omega$.

Since $E \geq 0$ (because $V(x) \geq 0$) we must have that, if $|E_0\rangle$ is the groundstate,

$$\hat{a}_-|E_0\rangle = 0 \quad \text{— (17)}$$

Then, only the eigenstates with energy $E_n = n\hbar\omega + E_0$, $\{|E_n\rangle\}$, $n = 0, 1, 2, 3, \ldots$ are allowed.

Hence the ladder operators give a unique set of eigenstates. (upto coeff.)

$$(17) \Rightarrow \quad \hat{a}_+ \hat{a}_- |E_0\rangle = 0 \quad \text{--- (18)}$$

$$(8) \Rightarrow \quad \left(\hat{H}/\hbar\omega - \frac{1}{2}\right)|E_0\rangle = 0 \quad \text{--- (19)}$$

$$\Rightarrow \quad \hat{H}|E_0\rangle = \frac{\hbar\omega}{2}|E_0\rangle \quad \text{--- (20)}$$

$$\text{or} \quad \boxed{E_0 = \frac{\hbar\omega}{2}} \quad \text{--- (21)}$$

$$\Rightarrow \quad \boxed{E_n = \left(n + \frac{1}{2}\right)\hbar\omega} \quad \text{--- (22)}$$

$$n = 0, 1, 2, \ldots$$

$E_0 \longrightarrow$ zero-point energy.