15/10/2025 Lec. 26. QM SHO - Ladder Operator Approach (ctd)

Since energies are indexed by n we can represent the eigenstates as

$$|E_n\rangle \equiv |n\rangle \quad \text{--- (1)}$$

$$\hat{H}|n\rangle = \left(n+\tfrac{1}{2}\right)\hbar\omega \quad \text{--- (2)}$$

$$\hat{N}|n\rangle = n|n\rangle \quad \text{--- (3)}$$

$n \rightarrow$ no. of quanta in the system

Let $$\hat{a}_+|n\rangle = C_n^+|n+1\rangle \quad \text{--- (4)}$$

$$\Rightarrow \langle n|\hat{a}_-\hat{a}_+|n\rangle = |C_n^+|^2 \quad \text{--- (5)}$$

i.e.

$$|C_n^+|^2 = \langle n|1+\underbrace{\hat{a}_+\hat{a}_-}_{\hat{N}}|n\rangle$$

$$= (1+n) \quad \text{--- (6)}$$

$$\Rightarrow |C_n^+| = \sqrt{n+1} \quad \text{--- (7)}$$

$$\& \quad \boxed{\hat{a}_+|n\rangle = \sqrt{n+1}\,|n+1\rangle} \quad \text{--- (8)}$$

III b) Let $\hat{a}_- |n\rangle = c_n^- |n-1\rangle$ — (9)

$$\Rightarrow \langle n | \hat{a}_+ \hat{a}_- | n\rangle = |c_n^-|^2$$

$$= n \quad \text{— (10)}$$

$$\therefore \quad \boxed{\hat{a}_- |n\rangle = \sqrt{n}\, |n-1\rangle} \quad \text{— (11)}$$

What about the wavefunctions?

Let $\psi_n(x) = \langle x | n \rangle$ — (12)

Consider

$$\hat{a}_- |0\rangle = 0 \quad \text{— (13)}$$

$$\Rightarrow 0 = \langle x | \hat{a}_- | 0\rangle = \frac{1}{\sqrt{2}} \langle x | (\hat{\xi} + i\hat{\pi}) | 0\rangle$$

$$= \frac{1}{\sqrt{2}} \left\{ \langle x | \hat{x} | 0\rangle \sqrt{\frac{m\omega}{\hbar}} + i \langle x | \hat{p} | 0 \rangle \frac{1}{\sqrt{m\omega\hbar}} \right\}$$

— (13)

$$\Rightarrow \sqrt{\frac{m\omega}{\hbar}}\, x\, \psi_0(x) + \frac{\hbar}{\sqrt{m\omega\hbar}} \frac{d\psi_0(x)}{dx} = 0 \quad - (14)$$

$$\text{or} \quad \frac{d\psi_0(x)}{dx} = -\left(\frac{m\omega}{\hbar}\right) x\, \psi_0(x) \quad - (15)$$

$$\Rightarrow \psi_0(x) = A_0 \exp\left(-\left(\frac{m\omega}{2\hbar}\right) x^2\right) \quad - (16)$$

Normalization

$$\int_{-\infty}^{\infty} dx\, |\psi_0(x)|^2 = 1$$

$$\Rightarrow |A_0|^2 \int_{-\infty}^{\infty} \exp\left(-\underbrace{\left(\frac{m\omega}{\hbar}\right)}_{\equiv \alpha} x^2\right) = 1$$

$$\Rightarrow |A_0|^2 = \sqrt{\frac{\alpha}{\pi}}$$

$$\Rightarrow |A_0| = \left(\sqrt{\frac{\alpha}{\pi}}\right)^{1/2} \quad - (17)$$

$$\therefore \quad \boxed{\psi_0(x) = \left(\frac{\alpha}{\pi}\right)^{1/4} \exp(-\alpha x^2/2) \qquad \alpha = \frac{m\omega}{\hbar}} \quad \text{—} (18)$$

For other eigenfunctions we just use the raising operator.

$$|1\rangle = \frac{1}{\sqrt{2}} \hat{a}_+ |0\rangle \quad \text{—} (19)$$

$\Rightarrow$

$$\psi_1(x) = \frac{1}{\sqrt{2}} \langle x | \hat{a}_+ | 0 \rangle$$

$$= \frac{1}{\sqrt{2}} \left(\sqrt{\frac{m\omega}{\hbar}}\, x\, \psi_0(x) - \frac{\hbar}{\sqrt{m\omega\hbar}} \psi_0'(x) \right)$$

$$= \frac{1}{\sqrt{2}} \left(\left(\frac{x}{x_0}\right) \psi_0(x) - \frac{d}{d(x/x_0)} \psi(x) \right) \quad \text{—} (20)$$

$\Rightarrow$ position space analogue of $\hat{a}_+$ is

$$\hat{a}_+ \longleftrightarrow \frac{1}{\sqrt{2}} \left(\xi' - \frac{d}{d\xi'} \right) \quad \text{—} (21)$$

$$\xi' = \sqrt{\frac{m\omega}{\hbar}}\, x.$$

Therefore the analog for $\hat{a}_+^n$ is

$$\hat{a}_+^n \longleftrightarrow \frac{1}{2^{n/2}}\left[\xi' - \frac{d}{d\xi'}\right]^n$$

$$\xi' = \sqrt{\frac{m\omega}{\hbar}}\, x \quad \text{— (21)}$$

This implies that:

$$\psi_n(x) = \frac{1}{2^{n/2}}\left[\xi' - \frac{d}{d\xi'}\right]^n \psi_0(x) \quad \text{— (22)}$$

$$= \frac{1}{2^{n/2}}\left(\frac{\alpha}{\pi}\right)^{1/4}\left[\xi' - \frac{d}{d\xi'}\right]^n e^{-\xi'^2/2} \quad \text{— (23)}$$

$$\alpha = 1/x_0^2 \quad \text{— (24)}$$

This is just another Rodrigues' formula for Hermite polynomials*

$$\psi_n(\xi) = \frac{1}{\sqrt{2^n n!}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} H_n(\xi)\, e^{-\frac{\xi^2}{2}}. \quad \text{— (25)}$$

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* (23) defines the probabilistic Hermite polynomials but we will use another one.

physicists' Hermite polynomials

$$
\left.\begin{aligned}
H_0(\xi) &= 1\\
H_1(\xi) &= 2\xi\\
H_2(\xi) &= 4\xi^2-2\\
H_3(\xi) &= 8x^3-12x\\
H_4(\xi) &= 16x^4-48x^2+12
\end{aligned}\right\} \quad (26)
$$

$\vdots$

Properties

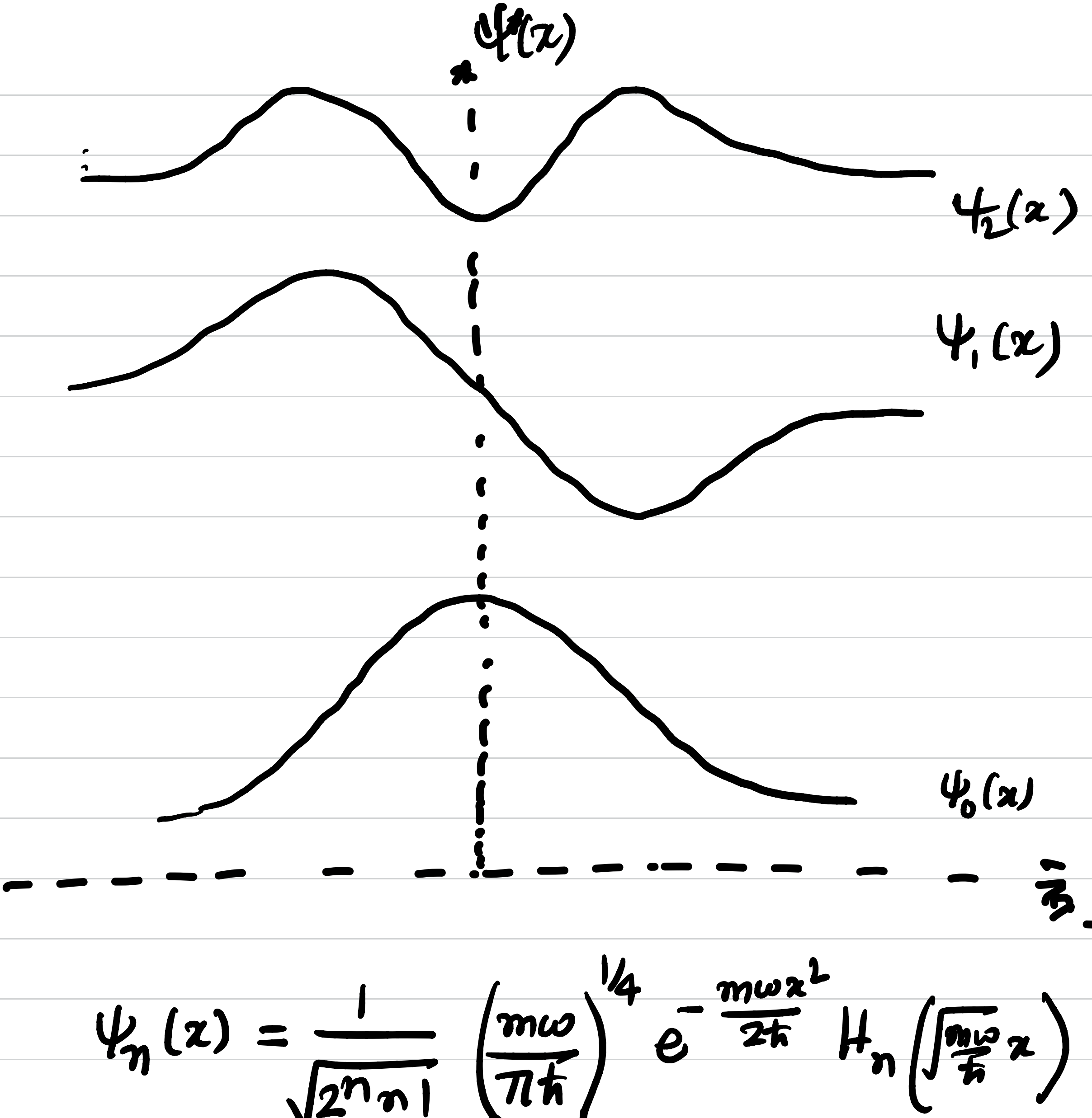
1. $H_n(-\xi) = H_n(\xi)(-1)^n$ parity (27)

2. $H_n(\xi)$ has n real roots
   $\Rightarrow \psi_n(x)$ has n nodes.

3. $$\int_{-\infty}^{\infty} H_n(\xi)\, H_m(\xi)\, e^{-\xi^2}\, d\xi$$

$$= 2^n n! \sqrt{\pi}\, \delta_{m,n} \quad (28)$$

4. $$H_{n+1}(x) = 2x H_n(x) - 2n H_{n-1}(x) \quad (29)$$



$$\psi_n(x) = \frac{1}{\sqrt{2^n n!}} \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega x^2}{2\hbar}} H_n\left(\sqrt{\frac{m\omega}{\hbar}}\, x\right) \quad (30)$$

$n = 0, 1, 2, \ldots$

$E_n = (n + \tfrac{1}{2})\hbar\omega$