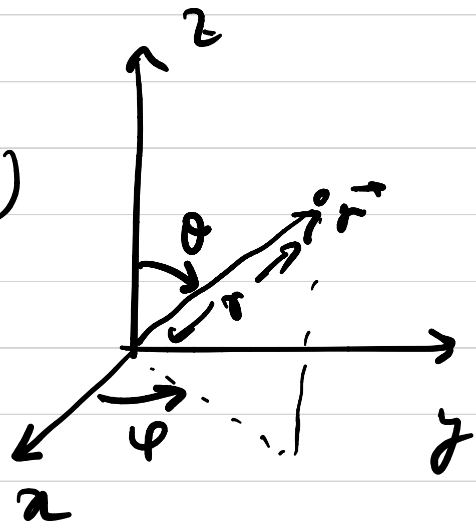
28/10/2025 Lecture 30. Angular Momentum

Particle in a central potential:

i.e. $V(\vec{r}) = V(x,y,z) = V(r)$ — (1)

Spherical
Polar coordinates



$(x,y,z) \rightarrow (r,\theta,\varphi)$

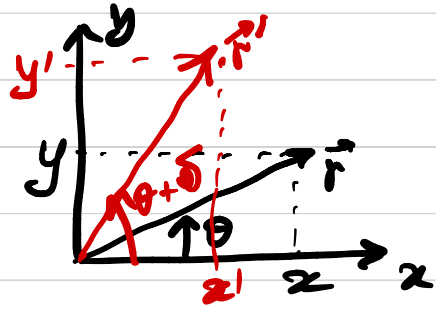
$r = [0,\infty)$

$\theta = [0,\pi]$

$\varphi = [0,2\pi]$ or $[-\pi,\pi]$

Counterclockwise rotation taken to be positive.

Rotation of xy-vector about z axis



Easy to see that

$$\left.\begin{aligned} x' &= x\cos\delta - y\sin\delta \\ y' &= x\sin\delta + y\cos\delta \\ z' &= z \end{aligned}\right\} \quad (2)$$

$$(x', y', z') \xleftarrow{R_z(\delta)} (x, y, z)$$

$$R_z(\delta) = \begin{pmatrix} \cos\delta & -\sin\delta & 0 \\ \sin\delta & \cos\delta & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (3)$$

$$\Rightarrow \vec{r}' = R_z(\delta)\,\vec{r} \quad (4)$$

lll^{ly},

$$\left.\begin{aligned} R_x(\delta) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\delta & -\sin\delta \\ 0 & \sin\delta & \cos\delta \end{pmatrix} \\ R_y(\delta) &= \begin{pmatrix} \cos\delta & 0 & \sin\delta \\ 0 & 1 & 0 \\ -\sin\delta & 0 & \cos\delta \end{pmatrix} \end{aligned}\right\} \quad (5)$$

An infinitesimal rotation by $\delta = \epsilon \to 0^+$ can be written approximately using

$$\left.\begin{aligned} \cos\epsilon &\simeq 1 - \epsilon^2/2 \\ \sin\epsilon &\simeq \epsilon \end{aligned}\right\} \quad (6)$$

Using these we can show, (H.W.) that

$$R_x(\epsilon)\, R_y(\epsilon) - R_y(\epsilon) R_x(\epsilon)$$

$$\simeq R_z(\epsilon^2) - 1 \quad — (7)$$

This is a commutation rule for rotations. Similar ones for other pairs of rotations can also be given.

In QM, such rotations would, in general, change the state of the system. Thus, we have a unique map from a starting state to the rotated state.

Let us define a rotation operator $\hat{D}(\hat{n},\varphi)$ which rotates about the $\hat{n}$ axis by an angle φ in the counter clockwise sense

If $\hat{n}$ is the z-axis then:

$$\hat{D}_z(\varphi)\,|x,y,z\rangle$$

$$\equiv |x',y',z'\rangle$$

$$= |x\cos\varphi - y\sin\varphi,\ x\sin\varphi + y\cos\varphi,\ z\rangle \quad \text{— (8)}$$

The general action of $\hat{D}(\hat{n},\varphi)$ on an arbitrary state $|\alpha\rangle$ is written as

$$\hat{D}(\hat{n},\varphi)\,|\alpha\rangle \equiv |\alpha'\rangle$$

$$= \int d^3r\,|x',y',z'\rangle\langle x,y,z|\alpha\rangle \quad \text{— (9)}$$

where $|x',y',z'\rangle$ are the rotated coordinates about $\hat{n}$ by φ.

Let us now consider an infinitesimal rotation about z by $\delta\varphi$

$$\hat{D}_z(\delta\varphi)\,|x,y,z\rangle \simeq |x - y\delta\varphi,\, x\delta\varphi + y, z\rangle$$

$$= \hat{\tau}_x(-y\delta\varphi)\,\hat{\tau}_y(x\delta\varphi)\,|x,y,z\rangle \quad \text{—(10)}$$

Using the definition of the translation operators $\hat{\tau}_\alpha(d\alpha)$.

Now, $\hat{\tau}_x(dx) = \hat{\mathbb{1}} - i\,dx\dfrac{\hat{p}_x}{\hbar}$ —(11)

$\therefore\ \hat{D}_z(\delta\varphi)\,|x,y,z\rangle$

$$= \left(\hat{\mathbb{1}} + i\hat{y}\delta\varphi\frac{\hat{p}_x}{\hbar}\right)\left(\hat{\mathbb{1}} - i\frac{\hat{x}\delta\varphi\hat{p}_y}{\hbar}\right)|x,y,z\rangle$$

$$= \left(\mathbb{1} + i\frac{\hat{y}\delta\varphi}{\hbar}\hat{p}_x - i\frac{\hat{x}\delta\varphi}{\hbar}\hat{p}_y\right)|x,y,z\rangle$$
$$+ O(\delta\varphi^2).$$

$$= \left[\hat{\mathbb{1}} - \frac{i}{\hbar}\delta\varphi\left(\hat{x}\hat{p}_y - \hat{y}\hat{p}_x\right)\right]|x\,y\,z\rangle \quad —(12)$$

$$\Rightarrow \hat{D}_z(\delta\varphi) = \hat{\mathbb{1}} - \frac{i}{\hbar}\delta\varphi\left(\hat{x}\hat{p}_y - \hat{y}\hat{p}_x\right) \quad —(13)$$

lll^{ly} we can show that

$$(14)\begin{cases}\hat{D}_y(\delta\varphi) = \hat{\mathbb{1}} + \frac{i}{\hbar}\delta\varphi\left(\hat{x}\hat{p}_z - \hat{z}\hat{p}_x\right)\\ \hat{D}_x(\delta\varphi) = \hat{\mathbb{1}} - \frac{i}{\hbar}\delta\varphi\left(\hat{y}\hat{p}_z - \hat{z}\hat{p}_y\right)\end{cases}$$

Or more generally we can write.

$$\hat{D}_{\hat{n}}(\delta\varphi) = \hat{\mathbb{1}} - \tfrac{i}{\hbar}\,\delta\varphi\,\hat{n}\cdot(\hat{\vec{r}}\times\hat{\vec{p}})$$

$$\equiv \hat{\mathbb{1}} - \tfrac{i}{\hbar}\,\delta\varphi\,\hat{n}\cdot\hat{\vec{J}} \quad \text{— (15)}$$

where $\hat{\vec{J}} = \hat{\vec{r}}\times\hat{\vec{p}}$ — (16)

is the angular momentum.

Orbital A.M.

In general, $\hat{D}(R)$ is a rotation operator corresponding to the rotation R. We require $\hat{D}(R)$ to completely mimic R.

Rotations	$\Longleftrightarrow$ Operators
Identity : $R\cdot\mathbb{1} = R$	$\hat{D}(R)\cdot\hat{\mathbb{1}} = \hat{D}(R)$
Closure : $R_1R_2 = R_3$	$\hat{D}(R_1)\hat{D}(R_2) = \hat{D}(R_3)$
Inverse : $RR^{-1} = \mathbb{I}$	$\hat{D}(R)\hat{D}^{-1}(R) = \hat{D}^{-1}\hat{D} = \hat{\mathbb{1}}$
Associativity : $R_1(R_2R_3) = (R_1R_2)R_3$	$\hat{D}(R_1)(\hat{D}(R_2)\hat{D}(R_3)) = (\hat{D}(R_1)\hat{D}(R_2))\hat{D}(R_3)$

With such an isomorphism defined between the 2 groups we can easily show that the rotation commutation relations in (17) imply that

$$[\hat{D}_x(\epsilon), \hat{D}_y(\epsilon)] = \hat{D}_z(\epsilon^2) - \hat{\mathbb{1}} \quad \text{— (18)}$$

$$\text{LHS} = \left[\left(\hat{\mathbb{1}} - i\frac{\epsilon}{\hbar}\hat{J}_x\right), \left(\hat{\mathbb{1}} - i\frac{\epsilon}{\hbar}J_y\right) \right]$$

$$= -\frac{\epsilon^2}{\hbar^2}[\hat{J}_x, \hat{J}_y] \quad \text{— (19)}$$

$$\text{RHS} = \hat{\mathbb{1}} - i\frac{\epsilon^2}{\hbar}\hat{J}_z - \hat{\mathbb{1}} = -i\frac{\epsilon^2}{\hbar}\hat{J}_z \quad \text{— (20)}$$

LHS = RHS $\Rightarrow$

$$\boxed{[\hat{J}_x, \hat{J}_y] = i\hbar J_z} \quad \text{— (21)}$$

The commutation relations for other rotation operator pairs also yield similar relations.

Generally,

$$\boxed{\left[\hat{J}_\alpha , \hat{J}_\beta\right] = i\hbar \sum_\gamma \epsilon_{\alpha\beta\gamma} \hat{J}_\gamma} \quad (12)$$

where α, β, γ run over cartesian components x, y, z.

$\epsilon_{\alpha\beta\gamma} \rightarrow$ Levi-Civitā tensor

$$= \begin{cases} 1 & \text{if } \alpha \neq \beta \neq \gamma \text{ and in cyclic order} \\ -1 & \text{if } \alpha \neq \beta \neq \gamma \text{ and not in cyclic order} \\ 0 & \text{if any 2 components are equal.} \end{cases}$$

$\Rightarrow \epsilon_{\alpha\beta\gamma} = -\epsilon_{\beta\alpha\gamma}$

Any Hermitian vector operator whose components satisfies the commutation algebra (22) will henceforth be called an <u>Angular Momentum Operator</u>

$$\hat{D}^{\dagger}(\hat{n},\delta\varphi) = \hat{1} + \frac{i}{\hbar}\hat{n}\cdot\vec{J}^{\dagger}\delta\varphi \quad \text{— (24)}$$

If $\hat{D}$ conserves norm then it is unitary. $\hat{D}^{\dagger} = \hat{D}^{-1}$ — (25)

$$\Rightarrow \quad \hat{1} + \frac{i}{\hbar}\delta\varphi\,\hat{n}\cdot\hat{\vec{J}}^{\dagger}$$

$$= \hat{1} + \frac{i}{\hbar}\delta\varphi\,\hat{n}\cdot\vec{J}$$

$$\Rightarrow \quad \hat{\vec{J}} = \hat{\vec{J}}^{\dagger} \Rightarrow \text{Hermitian} \quad \text{— (26)}$$

Note that while (22) is a general definition for an AM operator, the identification made in (16)

$$\hat{\vec{J}} = \hat{\vec{r}} \times \hat{\vec{p}}$$

is derived from its action on a position basis. In this case, we have the orbital angular momentum $\vec{L}$.

$\hat{\vec{L}} \rightarrow$ generator of infinitesimal spatial rotation.

Later on, we will also see another operator satisfying (22) but not operating on position bases

$\rightarrow$ Spin A.M.

As well as a total angular momentum operator.