APPENDIX TO LECTURE 31

UNITARY TRANSFORMATION OF STATES AND OPERATORS

A unitary transform effected by a unitary operator maps states from one basis set to another.

$$\hat{U} : \quad \hat{U}|\alpha_i\rangle = |\tilde{\alpha}_i\rangle \quad — (1)$$

where both $\{|\alpha_i\rangle\}$ & $\{|\tilde{\alpha}_i\rangle\}$ are a basis sets for the same Hilbert space.

$$\Rightarrow \sum_i |\alpha_i\rangle\langle\alpha_i| = \hat{\mathbb{1}} \quad — (2)$$

$$\sum_i |\tilde{\alpha}_i\rangle\langle\tilde{\alpha}_i| = \hat{\mathbb{1}} \quad — (3)$$

$$(1) \Rightarrow \quad \langle\alpha_j|\hat{U}|\alpha_i\rangle = \langle\alpha_j|\tilde{\alpha}_i\rangle = U_{ji} \quad — (4)$$

Consider

$$|\tilde{\Psi}\rangle = \hat{U}|\Psi\rangle = \sum_i \hat{U}|\alpha_i\rangle\langle\alpha_i|\Psi\rangle$$

$$= \sum_i |\tilde{\alpha}_i\rangle\langle\alpha_i|\Psi\rangle \quad — (5)$$

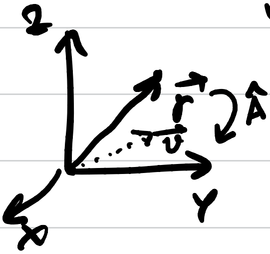
Also

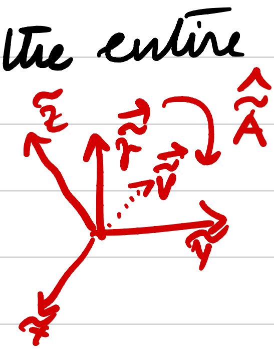
$$|\tilde{\Psi}\rangle = \sum_i |\tilde{\alpha}_i\rangle\langle\tilde{\alpha}_i|\tilde{\Psi}\rangle \quad — (6)$$

$$\Rightarrow \boxed{\langle\tilde{\alpha}_i|\tilde{\Psi}\rangle = \langle\alpha_i|\Psi\rangle} \quad — (7)$$

The coefficients of $|\Psi\rangle$ in the $|\alpha_i\rangle$ basis set are the same as the coefficient of $\hat{U}|\Psi\rangle$ in the $|\tilde{\alpha}_i\rangle$ basis.

This is equivalent to rotating the entire space the same way.





Now consider an operator $\hat{A}$.

Let $$|\Phi\rangle = \hat{A}|\Psi\rangle \quad \text{— (8)}$$

$$\left.\begin{aligned} |\tilde{\Phi}\rangle = \left(\hat{U}|\Phi\rangle\right) &= \hat{U}\hat{A}\hat{U}^{\dagger}\left(\hat{U}|\Psi\rangle\right) \\ &= \hat{\tilde{A}}|\tilde{\Psi}\rangle \end{aligned}\right\} \text{— (9)}$$

i.e. if $|\Psi\rangle \xrightarrow{\hat{A}} |\Phi\rangle$

then $|\tilde{\Psi}\rangle \xrightarrow{\hat{\tilde{A}} = \hat{U}\hat{A}\hat{U}^{\dagger}} |\tilde{\Phi}\rangle$ } (10)

30/10/2025 Lecture 31. Orbital Angular Momentum

$\vec{L}$: $$[\hat{L}_\alpha, \hat{L}_\beta] = i\hbar \sum_\gamma \epsilon_{\alpha\beta\gamma} \hat{L}_\gamma \quad -(1)$$

How do operators transform under spatial rotations?

Now, $$\hat{D}_z(\epsilon)\, \hat{x}\, |x,y,z\rangle = x\, |x+\epsilon y, y-\epsilon x, z\rangle \quad -(2)$$

$$(\hat{x} - \epsilon \hat{y})\, |x+\epsilon y, y-\epsilon x, z\rangle$$
$$\simeq x\, |x+\epsilon y, y-\epsilon x, z\rangle + O(\epsilon^2) \quad -(2')$$

Also, $$\hat{D}_z(\epsilon)\, \hat{x}\, |x\, y\, z\rangle = \hat{D}_z(\epsilon)\hat{x}\hat{D}_z^\dagger(\epsilon)\, \hat{D}_z(\epsilon)\, |x,y,z\rangle$$
$$\equiv \hat{\hat{x}}\, |x+\epsilon y, y-\epsilon y, z\rangle \quad -(3)$$

$$(\because \hat{D}_z^\dagger = \hat{D}_z^{-1})$$

Since (3) is true for any ~~ket~~ $|x,y,z\rangle$ & ϵ, we get

$$\boxed{\hat{D}_z(\epsilon)\, \hat{x}\, \hat{D}_z^\dagger(\epsilon) = \hat{x} - \epsilon \hat{y}} \quad -(4)$$
$$= \hat{\hat{x}}$$

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$$\boxed{\begin{aligned} \hat{\tilde{y}} &= \hat{D}_z(\epsilon)\,\hat{y}\,\hat{D}_z^{+}(\epsilon) = \hat{y} + \epsilon\,\hat{x} \\ \hat{\tilde{z}} &= \hat{D}_z(\epsilon)\,\hat{z}\,\hat{D}_z^{+}(\epsilon) = \hat{z} \end{aligned}} \quad \begin{aligned} &— (5) \\ &— (6) \end{aligned}$$

& The same relations hold for the components of the momentum generator $\vec{\hat{p}}$.

If $\vec{\tilde{r}} = (\hat{\tilde{x}}, \hat{\tilde{y}}, \hat{\tilde{z}})$ & $\vec{\hat{r}} \equiv (\hat{x}, \hat{y}, \hat{z})$

then $\vec{\hat{\tilde{r}}} \cdot \vec{\hat{\tilde{r}}} = \hat{\tilde{x}}^2 + \hat{\tilde{y}}^2 + \hat{\tilde{z}}^2$

$$= (\hat{x} - \epsilon\hat{y})^2 + (\hat{y} + \epsilon\hat{x})^2 + \hat{z}^2$$

$$= \hat{x}^2 + \hat{y}^2 + \hat{z}^2 + \epsilon^2(\hat{x}^2 + \hat{y}^2)$$

$$\simeq \hat{x}^2 + \hat{y}^2 + \hat{z}^2 + O(\epsilon^2)$$

$$= \vec{\hat{r}} \cdot \vec{\hat{r}} \quad — (7)$$

$\Rightarrow r^2 = \vec{\hat{r}} \cdot \vec{\hat{r}}$ remains invariant under the infinitesimal rotation

Same is also true of $\hat{p}^2 = \vec{\hat{p}} \cdot \vec{\hat{p}}$.

Consider the Hamiltonian of a central potential problem.

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{r}) \quad \text{— (8)}$$

$\hookrightarrow$ radial operator

$$\hat{D}_z(\epsilon)\,\hat{H}\,\hat{D}_z^{\dagger}(\epsilon) =$$

$$\frac{1}{2m}\hat{D}_z(\epsilon)\,\hat{p}^2\,\hat{D}_z^{\dagger}(\epsilon) + \hat{D}_z(\epsilon)\,V(\hat{r})\,\hat{D}_z^{\dagger}(\epsilon)$$

$$= \frac{\hat{p}^2}{2m} + V\left(\hat{D}_z(\epsilon)\,\hat{r}\,\hat{D}_z^{\dagger}(\epsilon)\right)$$

$$= \frac{\hat{p}^2}{2m} + V(\hat{r}) = \hat{H}. \quad \text{— (9)}$$

That is, the Hamiltonian is invariant under an infinitesimal z-rotation. For that matter, the Hamiltonian is invariant under an infinitesimal rotation about any axis. This is because we could choose any arbitrary direction as z-axis since the Hamiltonian itself has no preferred direction.

(9) $\Rightarrow$
$$\left(\hat{1} - \frac{i}{\hbar}\epsilon \hat{L}_z\right)\hat{H}\left(\hat{1} + \frac{i}{\hbar}\epsilon \hat{L}_z\right) = \hat{H}$$

$$\Rightarrow \quad \hat{H} - \frac{i}{\hbar}\epsilon\left[\hat{L}_z, \hat{H}\right] + O(\epsilon^2) = \hat{H}$$

$$\Rightarrow \quad \left[\hat{L}_z, \hat{H}\right] = 0 \qquad \text{— (10)}$$

(upto $O(\epsilon^2)$)

Since the rotational invariance is true of all axes, we also have

$$[\hat{\vec{L}}, \hat{H}] = 0 \quad — (11)$$

Then, the Ehrenfest theorem implies that

$$\frac{d\langle \vec{L} \rangle}{dt} = 0 \quad — (12)$$

and there are "good" quantum no.s. associated the angular momentum

That is $\vec{L}$ & $\hat{H}$ share a common set of eigenstates.

However, since the components of $\vec{L}$ do not mutually commute, only one component can be chosen to provide a quantum number simultaneously with energy eigenvalue.

Moreover,

$$[\hat{J}^2, \hat{J}_\alpha] = \sum_\beta [\hat{J}_\beta^2, \hat{J}_\alpha]$$

$$= \sum_{\beta} \left\{ \hat{J}_\beta [\hat{J}_\beta, \hat{J}_\alpha] + [\hat{J}_\beta, \hat{J}_\alpha] \hat{J}_\beta \right\}$$

$$= i\hbar \sum_{\beta} \sum_{\gamma} \epsilon_{\beta\alpha\gamma} \left(\hat{J}_\beta \hat{J}_\gamma + \hat{J}_\gamma \hat{J}_\beta \right)$$

$$= i\hbar \left[\sum_{\beta} \sum_{\gamma} \epsilon_{\beta\alpha\gamma} \hat{J}_\beta \hat{J}_\gamma + \sum_{\beta} \sum_{\gamma} \epsilon_{\gamma\alpha\beta} \hat{J}_\beta \hat{J}_\gamma \right]$$

(switched γ & β in the second term)

$$= i\hbar \sum_{\beta} \sum_{\gamma} \epsilon_{\beta\alpha\gamma} \left[\hat{J}_\beta \hat{J}_\gamma - \hat{J}_\beta \hat{J}_\gamma \right] = 0$$

$(\because \epsilon_{\gamma\alpha\beta} = -\epsilon_{\beta\alpha\gamma})$

i.e. $$\boxed{[\hat{J}^2, \hat{J}_\alpha] = 0} \quad \text{--- (13)}$$

i.e. Angular momentum magnitude square commutes with all components of $\vec{J}$.

So we can know the magnitude of the angular momentum along with one of its components simultaneously with arbitrary precision. Usually, the z-component is chosen.

For the orbital A.M. this means.

$$[\hat{L}^2, \hat{L}_\alpha] = 0 \qquad \alpha = x, y, z.$$

It's easy to show that since $[\hat{L}_\alpha, \hat{H}] = 0$ we must also have

$$[\hat{L}^2, \hat{H}] = \sum_\alpha [\hat{L}_\alpha^2, \hat{H}] = 0 \quad (14)$$

$\hat{L}^2$ & $\hat{L}_z$ share a common set of eigenstates. $\hat{L}_x$ & $\hat{L}_y$ remain uncertain

This situation is depicted in the figure →