4/11/2020 Lec 32. Eigenstates & Eigenvalues of Angular Momentum

The generalized angular momentum generator is defined through the following relations:

$$\hat{\vec{J}} = \hat{\vec{J}}^{\dagger} \quad \text{(Hermitian)} \qquad (1)$$

$$[\hat{J}^2, \hat{J}_\alpha] = 0 \qquad \alpha = x, y, z \qquad (2)$$

$$[\hat{J}_\alpha, \hat{J}_\beta] = i\hbar \sum_\gamma \epsilon_{\alpha\beta\gamma} \hat{J}_\gamma \qquad (3)$$

Here $\hat{J}^2 = \hat{\vec{J}} \cdot \hat{\vec{J}} = \hat{J}_x^2 + \hat{J}_y^2 + \hat{J}_z^2$ — (4)

(2),(3) $\Rightarrow$ that we can simultaneously determine J^2 & only one of the components of $\vec{J}$. Conventionally, the $\hat{J}_z$ component is quantized simultaneously with $\hat{J}^2$.

Let's look for simultaneous eigenkets of $\hat{J}^2, \hat{J}_z$. Let's call these as $\{|\lambda, m_j\rangle\}$ such that

$$\hat{J}_z |\lambda, m_j\rangle \equiv m_j \hbar |\lambda, m_j\rangle \quad \text{—(5)}$$

&

$$\hat{J}^2 |\lambda, m_j\rangle \equiv f(\lambda, m_j)\hbar^2 |\lambda, m_j\rangle \quad \text{—(6)}$$

Note that → f, m_j are real $\because \hat{\vec{J}}$ is Hermitian

→ f, m_j are unitless

→ since $m_j\hbar$ is only a component of $\vec{J}$

$$\boxed{f - m_j^2 \geq 0} \quad \text{—(7)}$$

Define $\hat{J}_\pm = \hat{J}_x \pm i\hat{J}_y$ —(8)

$$\Rightarrow \quad \hat{J}_x = \frac{\hat{J}_+ + \hat{J}_-}{2} \quad \text{—(9)}$$

$$\hat{J}_y = \frac{\hat{J}_+ - \hat{J}_-}{2i} \quad \text{—(10)}$$

These satisfy:

$$[\hat{J}_z, \hat{J}_\pm] = \pm\hbar\hat{J}_\pm \quad \text{—(11)}$$

$$[\hat{J}_+, \hat{J}_-] = 2\hbar\hat{J}_z \quad \text{—(12)}$$

$$[\hat{J}^2, \hat{J}_\pm] = 0 \quad \text{—(13)}$$

(13) $\Rightarrow \hat{J}^2\left(\hat{J}_\pm |\lambda, m_j\rangle\right) = f(\lambda, m_j)\hbar^2 \left(\hat{J}_\pm |\lambda, m_j\rangle\right)$ — (14)

i.e. $\hat{J}_\pm |\lambda, m_j\rangle$ are also eigenstates of $\hat{J}^2$ with the same eigenvalue, $f(\lambda, m_j)$, as $|\lambda, m_j\rangle$

Now, $\hat{J}_z\left(\hat{J}_+ |\lambda, m_j\rangle\right)$

$$= \left(\hat{J}_+ \hat{J}_z + \hbar \hat{J}_+\right)|\lambda, m_j\rangle$$

(using (11))

$$= (m_j+1)\hbar\left(\hat{J}_+ |\lambda, m_j\rangle\right) \quad \text{— (15)}$$

$\Rightarrow$ $\hat{J}_+ |\lambda, m_j\rangle$ is also an eigenstate of $\hat{J}_z$ with eigenvalue $(m_j+1)\hbar$.

i.e. $\hat{J}_+$ raises the z-component of $\vec{J}$ by $\hbar$.

|||ly $\hat{J}_z\left(\hat{J}_-|\lambda, m_j\rangle\right) = (m_j - 1)\hbar\left(\hat{J}_-|\lambda, m_j\rangle\right)$ — (16)

i.e. $\hat{J}_-$ lowers the z-component by $\hbar$.

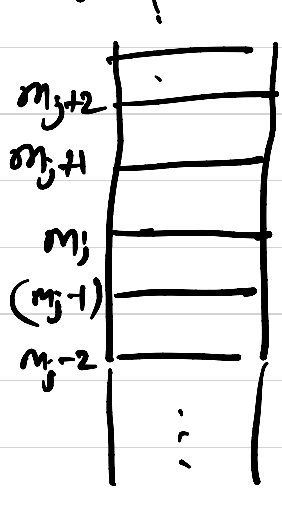
i.e. $$\hat{J}_\pm|\lambda, m_j\rangle \propto |\lambda, m_j \pm 1\rangle \qquad (17)$$

where $\{|\lambda, m_j\rangle\}$ are taken to be normalized.

In general, $$\left(\hat{J}_\pm\right)^n|\lambda, m_j\rangle \propto |\lambda, m_j \pm n\rangle \qquad (18)$$

(18) defines a ladder of states, originating from a given m_j.

All the states have the same eigenvalue for $\hat{J}^2$, i.e. $f(\lambda, m_j)$



But, since all $\hat{J}_z$ eigenvalues must satisfy (7) for a given f., we must have a maximum m_j^2 value. Let us refer to this value a j^2:

$$j^2 = \max\{m_j^2\}\Big|_{\text{for a given } \lambda} \quad \text{— (19)}$$

$$\Rightarrow \quad -j \leq m_j \leq j \quad \text{for a given } \lambda \quad \text{— (20)}$$

To ensure this we must impose.

$$\hat{J}_+ |\lambda, j\rangle = 0 \quad \text{— (21)}$$

$$\hat{J}_- |\lambda, -j\rangle = 0 \quad \text{— (22)}$$

(Compare this with the boundary conditions on the QHO operators)

From (21) we have,

$$\hat{J}_- \hat{J}_+ |\lambda, j\rangle = 0 \quad \text{— (23)}$$

$$\hat{J}_-\hat{J}_+ = (\hat{J}_x - i\hat{J}_y)(\hat{J}_x + i\hat{J}_y)$$

$$= \hat{J}_x^2 + \hat{J}_y^2 + i[\hat{J}_x, \hat{J}_y]$$

$$= \hat{J}^2 - \hat{J}_z^2 - \hbar \hat{J}_z \quad — (24)$$

(using (3), (4) & (8))

Substituting (24) in (23) we get

$$f\hbar^2 - j^2\hbar^2 - j\hbar^2 = 0$$

or $$f(\lambda, j) = j(j+1) \quad — (25)$$

f is the magnitude of $\vec{J}$ and remains the same for all m_j. Therefore, we can replace the quantum number λ by j which fully determines f.

$$\therefore \quad \boxed{\hat{J}^2 |j, m_j\rangle = j(j+1)\hbar^2 |j, m_j\rangle} \quad — (26)$$

Starting from $|j, j\rangle$ and repeatedly applying $\hat{J}_-$ we get the state

$$\{ |j, j-1\rangle, |j, j-2\rangle, \ldots, |j, m_j\rangle \ldots |j, -j\rangle \} \quad (27)$$

Thus, the only values of m_j allowed for a given j.

Since this implies that for each j we have $n = 2j+1$ m_j values we have that

$$j = \frac{(n-1)}{2} \quad \text{where } n \text{ is positive integer}$$

$$\Rightarrow j = 0, \tfrac{1}{2}, 1, \tfrac{3}{2}, 2, \tfrac{5}{2}, \ldots \quad (28)$$

$\therefore$ j can have integral or half-integral values.

$j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$

$m_j = -j, \ldots, j$

$$\boxed{\begin{aligned} \hat{J}^2 |j, m_j\rangle &= j(j+1)\hbar^2 |j, m_j\rangle \\ \hat{J}_z |j, m_j\rangle &= m_j \hbar\, |j, m_j\rangle \end{aligned}} \quad (29)$$

An infinitesimal spatial rotation of a position ket can be achieved using

$$\hat{D}_z(\epsilon) = \hat{\mathbb{1}} - i\frac{\epsilon}{\hbar}\hat{L}_z \quad \text{— (30)}$$

where $\hat{L}_z$ is the orbital A.M.

$$\hat{D}_z(\epsilon)|r,\theta,\varphi\rangle = |r,\theta,\varphi+\epsilon\rangle \quad \text{— (31)}$$

where we have used spherical polar coordinates for the position kets.

Now, a rotation by an angle α can be achieved in steps as.

$$\hat{D}_z(\alpha) = \left(\hat{D}_z(\alpha/N)\right)^N \quad \text{— (32)}$$

$$N \in \mathbb{Z}$$

If $N \to \infty$, then

$$\lim_{N\to\infty}\left(\hat{D}_z(\alpha/N)\right)^N = \lim_{N\to\infty}\left(\mathbb{1} - i\left(\frac{\alpha}{N}\right)\frac{1}{\hbar}\hat{L}_z\right)^N$$

$$= \exp\left(-i\frac{\alpha}{\hbar}\hat{L}_z\right) \quad \text{— (33)}$$

where we have used the approximation to the exponential:

$$e^{-x} \simeq \lim_{N\to\infty}\left(1-\frac{x}{N}\right)^N \quad — (34)$$

$$\therefore \quad \boxed{\hat{D}_z(\alpha) = \exp\left(-i\frac{\alpha}{\hbar}\hat{L}_z\right)} \quad — (35)$$

Now,

$$\hat{D}_z(\alpha)\,|r,\theta,\varphi\rangle = |r,\theta,\varphi+\alpha\rangle \quad — (36)$$

$$\Rightarrow \quad \exp\left(-i\frac{\alpha}{\hbar}\hat{L}_z\right)|r,\theta,\varphi\rangle = |r,\theta,\varphi+\alpha\rangle \quad — (37)$$

Let $|l,m_l\rangle$ be orbital A.M. eigenstates so that

$$\hat{L}_z\,|l,m_l\rangle = m_l\hbar\,|l,m_l\rangle \quad — (38)$$

$$\hat{L}^2\,|l,m_l\rangle = l(l+1)\hbar^2\,|l,m_l\rangle \quad — (39)$$

$$\Rightarrow \quad \langle l,m_l|\hat{D}_z(\alpha)|r,\theta,\varphi\rangle = \langle l,m_l|r,\theta,\varphi+\alpha\rangle \quad — (40)$$

From (37) we have also that

$$\langle l, m_l \mid r, \theta, \varphi + \alpha \rangle = \exp(-i\alpha m_l) \langle l, m_l \mid r, \theta, \varphi \rangle \quad -(41)$$

For $\alpha = 2\pi$ we get

$$\langle l, m_l \mid r, \theta, \varphi + 2\pi \rangle = e^{-i 2\pi m_l} \langle l, m_l \mid r, \theta, \varphi \rangle \quad -(42)$$

Now, $\langle l\, m_l \mid r\, \theta\, \varphi \rangle \equiv \psi_{l m_l}(r, \theta, \varphi)$ $-(43)$

is the wavefunction corresponding to $|l, m_l\rangle$. Since the wavefunction must be single-valued, and since a 2π rotation leaves the position invariant we must have:

$$e^{-i 2\pi m_l} = 1 \quad -(44)$$

or $m_l \in \mathbb{Z}$ $-(45)$

Since, m_ℓ is among $\{-\ell, -\ell+1, \ldots \ell\}$,

(45) $\Rightarrow$ $\ell \in \mathbb{Z}$.

Therefore, the integral values of j are attained for A.M. describing spatial rotations of position kets.

Orbital AM has only integral ℓ values possible.

$$\left.\begin{array}{l} \ell = 0, 1, 2, \ldots \\ m_\ell = -\ell, -\ell+1, \ldots, \ell-1, \ell. \end{array}\right\} \quad (46)$$