10/11/2025

# Lecture 33: Differential eqn & eigenfunction for orbital A.M.

## Matrix elements of angular momentum ladder operators

Let $\hat{J}_{\pm} |j, m_j\rangle \equiv c^{\pm}_{j,m_j} \hbar \, |j, m_j \pm 1\rangle$ — (1)

$$\Rightarrow \hbar^2 |c^{+}_{j,m_j}|^2 = \langle j, m_j | \hat{J}_- \hat{J}_+ | j, m_j \rangle \quad —(2)$$

where we have used

$$\hat{J}_- = \hat{J}_+^{\dagger} \quad —(3)$$

We saw that $\hat{J}_- \hat{J}_+ = \hat{J}^2 - \hat{J}_z^2 - \hbar \hat{J}_z$ — (4)

$$\therefore \hbar^2 |c^{+}_{j,m_j}|^2 = \langle j, m_j | \hat{J}^2 - \hat{J}_z^2 - \hbar \hat{J}_z | j, m_j \rangle$$

$$\Rightarrow |c^{+}_{j,m_j}|^2 = \left\{ j(j+1) - m_j^2 - m_j \right\}$$

$$\Rightarrow c^{+}_{j,m_j} = \sqrt{j(j+1) - m_j(m_j+1)} \quad —(5)$$

$|||^{ly}$ we can show that

$$c^{-}_{j,m_j} = \sqrt{j(j+1) - m_j(m_j - 1)} \qquad \text{— (5)}$$

$\therefore$

$$\boxed{\hat{J}_{\pm} |j, m_j\rangle = \hbar \sqrt{j(j+1) - m_j(m_j \pm 1)}\, |j, m_j \pm 1\rangle} \qquad \text{— (6)}$$

Since $\{|j, m_j\rangle\}$ are simultaneous eigenkets of $\hat{J}^2$ & $\hat{J}_z$ (both Hermitian) we have that

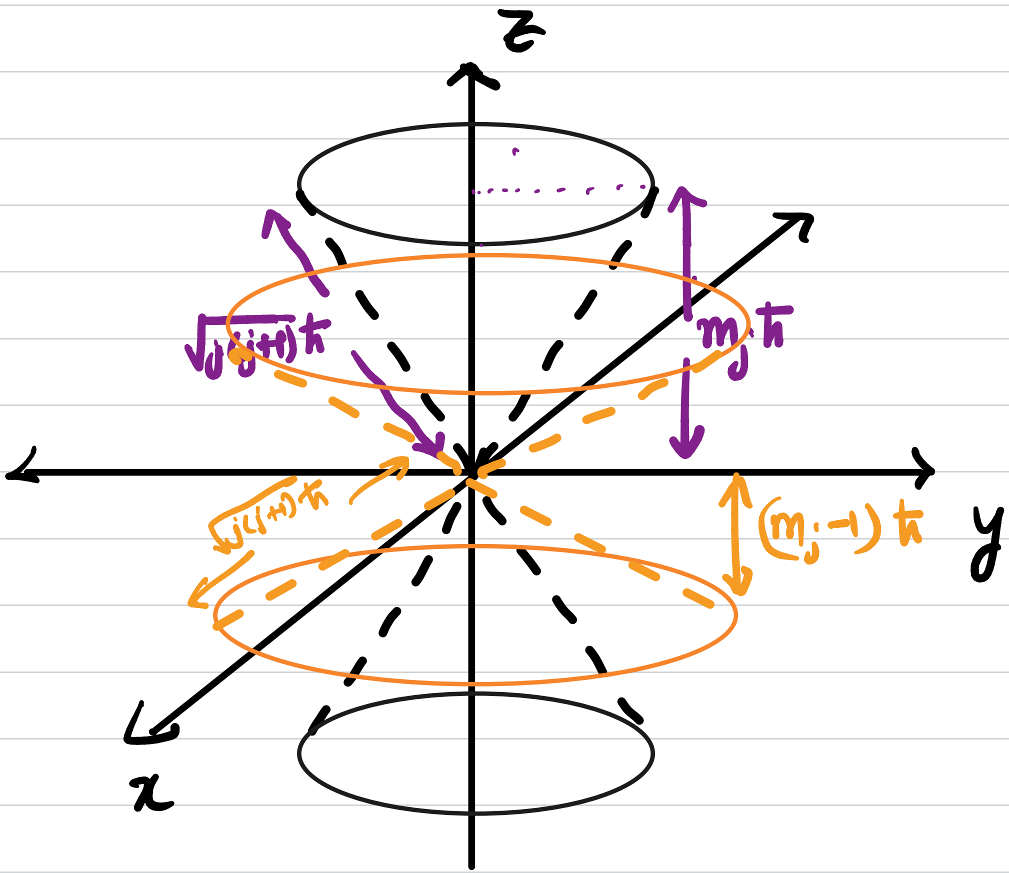
$$\boxed{\langle j', m_j' | j, m_j\rangle = \delta_{jj'} \delta_{m_j m_j'}} \qquad \text{— (7)}$$

Noting this, we can write down

$$\langle j' m_j' | \hat{J}_{\pm} | j\, m_j\rangle \qquad \text{— (8)}$$

$$= \left(\hbar \sqrt{j(j+1) - m_j(m_j \pm 1)}\right) \delta_{j',j \pm 1}\, \delta_{m_j', m_j \pm 1}$$

The A.M. vector lies on the surface of a cone about the z-axis.



## Eigenvalue problem of $\hat{L}_z$:

Now, $$\hat{D}_z(\epsilon) = \hat{1} - \frac{i}{\hbar}\epsilon \hat{L}_z \quad |\epsilon| \to 0^+ \qquad (9)$$

Consider position kets in spherical polar representation: $\{|r,\theta,\varphi\rangle\}$

$$\hat{D}_z(\epsilon)\,|r,\theta,\varphi\rangle$$
$$= |r,\theta,\varphi+\epsilon\rangle \qquad (10)$$

$\because$ $\hat{D}_z^{\dagger}(\epsilon) = \hat{D}_z(-\epsilon)$ (unitarity) $\qquad (11)$

we have
$$\langle r\theta\varphi|\,\hat{D}_z(\epsilon)$$
$$= \left(\hat{D}_z^{\dagger}(\epsilon)\,|r\theta\varphi\rangle\right)^{\dagger}$$
$$= \langle r,\theta,\varphi-\epsilon| \qquad (12)$$

$$\Rightarrow \quad \langle r\,\theta\,\varphi | \hat{D}_z(\epsilon) | \Psi \rangle \qquad \text{(arbitrary } |\Psi\rangle)$$

$$= \langle r, \theta, \varphi - \epsilon | \Psi \rangle = \Psi(r, \theta, \varphi - \epsilon)$$

$$\simeq \Psi(r\theta\varphi) - \epsilon \frac{\partial \Psi(r,\theta,\varphi)}{\partial \varphi} \quad \text{— (13)}$$

from (9) we also have

$$\langle r,\theta,\varphi | \hat{D}_z(\epsilon) | \Psi \rangle$$

$$= \langle r\theta\varphi | \left(\hat{1} - \frac{i}{\hbar}\epsilon \hat{L}_z\right) | \Psi \rangle$$

$$= \Psi(r,\theta,\varphi) - \frac{i}{\hbar}\epsilon \langle r\theta\varphi | \hat{L}_z | \Psi \rangle \quad \text{— (14)}$$

Comparing (13) & (14) we have

$$\boxed{\langle r,\theta,\varphi | \hat{L}_z | \Psi \rangle = -i\hbar \frac{\partial}{\partial \varphi} \Psi(r,\theta,\varphi)} \quad \text{— (15)}$$

Eq. (15) is the position space representation of the action of $\hat{L}_z$ on a state $|\Psi\rangle$

$$\Psi(r,\theta,\varphi) \xrightarrow{\hat{L}_z} -i\hbar \frac{\partial}{\partial\varphi}\Psi(r,\theta,\varphi) \quad -(16)$$

$\therefore$ If $|\Psi\rangle = |l\, m_l\rangle$ (orbital AM eigen state)

& $\Psi_{lm_l}(r\theta\varphi)$ is the corresponding wavefunction then we get from (15)

$$-i\hbar \frac{\partial}{\partial\varphi}\Psi_{lm_l}(r\theta\varphi) = m_l \hbar\, \Psi_{lm_l}(r\theta\varphi) \quad -(17)$$

Since (17) only determines $\varphi$ dependence we can write

$$\Psi_{lm_l}(r\theta\varphi) = f_{lm_l}(r,\theta)\,\Phi_{m_l}(\varphi) \quad -(18)$$

$\therefore$ we have :

$$\frac{d\Phi_{m_l}(\varphi)}{d\varphi} = i m_l \Phi_{m_l}(\varphi) \quad — (19)$$

$$\Rightarrow \quad \Phi_{m_l}(\varphi) = A e^{i m_l \varphi} \quad - (20)$$

$m_l = 0, \pm 1, \pm 2, \cdots \pm l$

A is determined by normalization

$$\langle \Phi_{m_l} | \Phi_{m_l} \rangle = |A|^2 \int_0^{2\pi} d\varphi \, |e^{i m_l \varphi}|^2$$

$$= |A|^2 \cdot 2\pi = 1 \quad — (21)$$

$$\Rightarrow \quad |A| = A = 1/\sqrt{2\pi} \quad - (22)$$

$$\therefore \quad \boxed{\Phi_{m_l}(\varphi) = \frac{1}{\sqrt{2\pi}} e^{i m_l \varphi}} \quad — (22)$$

$m_l = 0, \pm 1, \pm 2, \cdots \pm l$

Note that, by itself, (19) doesn't quantize $m_\ell$. The single-valuedness of $\Phi_{m_\ell}$ will restrict $m_\ell$ to integral values. The $\hat{L}^2$ equation will further restrict $|m_\ell| \leq \ell$. We have used this information from the algebra done before.

A simple model problem where (19) comes in handy, is that of a particle on a ring of radius $R$.

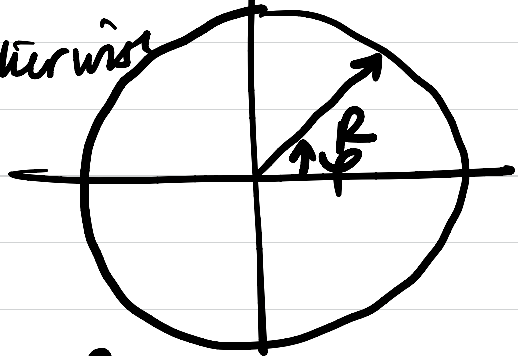
If the ring is in the $xy$ plane, then the only angular momentum is $L_z$.

The classical Hamiltonian for this problem is

$$H = \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2\right) \quad \text{——} \quad (24)$$

$$\dot{x} = dx/dt\,, \quad \dot{y} = \frac{dy}{dt}$$

$$V(r,\theta,\varphi) = \begin{cases} 0 & r = R,\ \theta = \pi/2 \\ \infty & \text{otherwise} \end{cases} \quad —(25)$$




$x = R\cos\varphi$

$y = R\sin\varphi$

$$\Rightarrow \quad \dot{x} = -R\sin\varphi\,\dot{\varphi}$$
$$\dot{y} = R\cos\varphi\,\dot{\varphi}$$

$$\& \quad H = \frac{m}{2}R^2(\sin^2\varphi + \cos^2\varphi)\dot{\varphi}^2$$

$$= \frac{mR^2\dot{\varphi}^2}{2} = \frac{mR^2\omega^2}{2} \quad —(26)$$

$\dot{\varphi} \rightarrow$ angular velocity $\omega$

rotational velocity $\quad v = R\omega \quad —(27)$

$$\therefore \quad H = \frac{mv^2}{2} = \frac{(mvR)^2}{2mR^2}$$

$$= L_z^2/2I \quad —(28)$$

where $L_z = mvR$

$I = mR^2$ (moment of inertia)

Thus, the quantum counterpart of this Hamiltonian is

$$\hat{H} = \hat{L}_z^2 / 2I \quad — (29)$$

$$\Rightarrow [\hat{L}_z, \hat{H}] = 0 \quad — (30)$$

Thus, they have simultaneous eigenstates. $\therefore$ eigenstates of $\hat{H}$ are $\{|m_\ell\rangle\}$ $m_\ell = 0, \pm 1, \pm 2, \ldots$

$$\Rightarrow \hat{H}|m_\ell\rangle = \frac{\hat{L}_z^2}{2I}|m_\ell\rangle$$

$$= \frac{m_\ell^2 \hbar^2}{2I}|m_\ell\rangle \quad — (31)$$

$$\Rightarrow \quad E_{m_l} = \frac{m_l^2 \hbar^2}{2I} \qquad m_l = 0, \pm 1, \pm 2, \ldots \qquad \text{— (32)}$$

are the allowed energy eigenvalues

Note that here $\vec{L} = \hat{L}_z \hat{k}$.

$\therefore$ there is no restriction of $m_l$ other than the integral values.

Eigenfunction are $\Phi_{m_l}(\varphi) = \frac{1}{\sqrt{2\pi}} e^{i m_l \varphi}$ — (33)

Can show that (HW)