12/8/2025 Day 5. **Mathematical Structure of QM states: Vector Spaces intro**

From the 2-slit example we also note that the $\bar{e}$'s emerging at the detector were associated with a wavefunction ψ which itself resulted from the superposition of the waves ψ_1 & ψ_2 from the 2 slits.

i.e. $$\psi(\vec{r},t) = \psi_1(\vec{r},t) + \psi_2(\vec{r},t) \quad \text{— (1)}$$

It's easy to convince ourselves that, through some combination of slits & sources, one can always represent any state of the electron in terms of superposition of other state.
This superposition principle is then a feature of QM states.

Furthermore, if the wavefunction is multiplied by a constant (say, A), then the probability distribution implied by the state remains unchanged. Thus, the states are determined by the wavefunctions upto an arbitrary constant multiple.

These observations can be bundled into our first postulate.

Postulate 1: Each state of a dynamical system, at a particular time, corresponds to a wave function, say Ψ. The correspondence is such that, if a state results from superposition of certain other states, the wavefunction is linearly expressible in terms of the corresponding wavefunctions of the other states, and conversely.

$$\Psi = c_1 \Psi_1 + c_2 \Psi_2 \quad \text{—— (2)}$$

where $c_1, c_2 \in \mathbb{C}$ complex number in general.

From the slit-based thought expts., we also anticipate that the order in which we combine these states does not matter and all should yield the same (superposed) state.

i.e.

$$\Psi = \psi_1 + \psi_2 = \psi_2 + \psi_1 \qquad (3)$$

and

$$\Psi = \psi_1 + (\psi_2 + \psi_3) = (\psi_1 + \psi_2) + \psi_3 \qquad (4)$$

We also consider, Ψ & $c\Psi$, where c is an arbitrary complex number (non-zero) are corresponding to the same state.

We note that there is nothing special about a probability distribution in real space. We could just as well chosen to measure the momentum distribution of the $\bar{e}$s. This would have let to a momentum dependent wavefunction $\Phi(\vec{p}, t)$.

For that matter we could have measured the distribution of any dynamical property and associated a wave function with. All these would, of course, describe the same state (same preparation of system). This means that there is an object, more fundamental than any of the wave function, that identifies the state. Let's denote this object as $|\Psi\rangle$, where $|\,\rangle$ is called a ket, and Ψ inside reminds us that this state is associated with the wavefunction $\Psi(\vec{r},t)$.

We can recast everything we are about to discuss in terms of these kets with the equivalence $|\Psi\rangle \longleftrightarrow \Psi(\vec{r},t)$. But for now let's just stick to wavefunctions

The properties of QM states listed out in postulate 1 are reminiscent of a mathematical construct called a vector space which we will revise now.

Vector Spaces: A collection of objects, $\mathbb{V}$, $|1\rangle, |2\rangle, |3\rangle, \ldots, |W\rangle$ (the ket symbol being a personal choice not to be confused with the QM state), for which there exists,

a) A definite rule for combining two objects (sum/addition), denoted $|V\rangle + |W\rangle$

b) A definite rule for multiplication by "scalars" $a, b, \ldots$, denoted $a|V\rangle$.

is said to form a vector space iff.

— P.T.O —

1. The result of each of the operations above is an object in the same collection. [Closure]

$$\underbrace{a|V\rangle + b|W\rangle}_{\text{Vector sum}} \in \mathbb{V}$$

for $a, b \rightarrow$ scalars real or complex.

2. Scalar multiplication is

(a) distributive in the vectors

$$a(|V\rangle + |W\rangle) = a|V\rangle + a|W\rangle$$

↳ 2 ways of forming vectors are equivalent

(b) distributive in scalars

$$(a+b)|V\rangle = a|V\rangle + b|V\rangle$$

(c) associative $\quad a(b|V\rangle) = (ab)|V\rangle$

3. (Vector) addition is

(a) commutative

$$|v\rangle + |w\rangle = |w\rangle + |v\rangle$$

(b) associative

$$|v\rangle + (|w\rangle + |x\rangle)$$
$$= (|v\rangle + |w\rangle) + |x\rangle$$

4. There exists a null object (vector), $|\emptyset\rangle$ in $\mathbb{V}$ obeying

$$|v\rangle + |\emptyset\rangle = |v\rangle \quad \forall\, |v\rangle \in \mathbb{V}$$

5. For every $|V\rangle \in \mathbb{V}$, there exists an inverse $|-V\rangle \in \mathbb{V}$, such that

$$|v\rangle + |-v\rangle = |\emptyset\rangle$$

This just implies:

$$|-V\rangle = |\emptyset\rangle + (-1)|V\rangle$$
$$= (-1)|V\rangle.$$

These axioms define the objects as "vectors" in their "Vector space".

They imply (show):

→ $|0\rangle$ is unique. i.e. if a $|0'\rangle$ satisfies the same properties as $|0\rangle$ then $|0\rangle \equiv |0'\rangle$

→ $0|V\rangle = |0\rangle$

→ $|-V\rangle = -|V\rangle$ (see above)

→ $|-V\rangle$ is, hence, unique additive inverse.

Examples:

a) Cartesian vectors in 3-d space

b) Set of all 2×2 matrices

c) Set of all functions $f(x)$, $0 \le x \le L$ such that $f(x=0) = f(x=L) = 0$

Most importantly, we recognize that QM wavefunctions also form a vector space !!