13/8/2025 Day 6: Inner Product Spaces

Linear independence: A set of vectors ($\in \mathbb{V}$)

$|1\rangle, |2\rangle, |3\rangle, \ldots, |n\rangle$ are said to be linearly independent iff the only solution to the equation

$$\sum_{i=1}^{n} a_i |i\rangle = |0\rangle \quad \forall a_i \in \mathbb{C} \qquad — (5)$$

is $a_1 = a_2 = \cdots a_n = 0$.

The main consequence of this is that we cannot construct any vector in this set by an addition of other vectors in the set.

If there is a non-trivial solution, say for a_3 & a_4, then we can write

$$|3\rangle = -\frac{a_4}{a_1}|4\rangle \qquad — (6)$$

Then $|3\rangle$ is said to linearly dependent on $|4\rangle$.

Eg. 2 Cartesian vectors in a plane are linearly independent as long as they are not parallel.

A 3rd vector, when defined, would necessarily be linearly dependent on these two.

For any vector space one can determine the maximum number of linearly independent (L.I.) vectors that can be defined.

<u>Theorem 1</u>: The largest number of L.I. vectors possible in a vector space V is called the dimension of the space.

So, if there are at most N L.I. vectors that can be defined in $\mathbb{V}$, then we denote the space as $\mathbb{V}^N$.

e.g. $\mathbb{R}^2 \leftrightarrow$ vector space (V.S.) formed by pairs of real numbers.

This is the same as the 2-D cartesian space we just spoke of. (Why?)

Given such a set of L.I vectors $\{|u_i\rangle\}_{i=1,N}$ in $\mathbb{V}^N$, this means that any other vector $|V\rangle \in \mathbb{V}$ is linearly dependent on them.

That is,

$$\boxed{|V\rangle = \sum_{i=1}^{N} v_i |u_i\rangle} \quad — (6)$$

where $v_i \in \mathbb{C}$ for a complex V.S.
or $\in \mathbb{R}$ for a real V.S.

$\{v_i\}_{i=1,N}$ are called the "components" of $|V\rangle$ in terms of $\{|u_i\rangle\}_{i=1,N}$

(6) is also termed as a resolution of the vector $|V\rangle$ or as a representation of $|V\rangle$ in terms of the $\{|u_i\rangle\}$.

N.B. 1) The largest L.I. set of vectors is not unique.

2) For a given choice of $(L.I)^N$ the components of a vector $|V\rangle$, $\{v_i\}$, are, however, unique.

There are definitions important to mention.

<u>Span</u>: A set of vectors in V^N $\{|w_i\rangle, i = 1, L\}$ such that any vector in V^N can be represented in terms of them is called a "span" of V^N and is said to span the space. Note, there's no necessity that they be L.I.

Basis set: A set of linearly independent vectors $\{|u_i\rangle i=1,N\}$ which span V^N is said to be a basis set of the vector space.

Q. Do we expect L to be larger or smaller than N?

Q. Is a basis set for a space unique?

This seems like a useful idea. But, how do we determine $\{v_i\}$?

In Cartesian space, if

$$\vec{V} = v_1 \vec{u}_1 + v_2 \vec{u}_2 \quad \text{— (7)}$$

then, we can use dot-products to write down.

- P.T.O -

$$(8)\begin{cases} (\vec{u}_1 \cdot \vec{V}) = v_1 (\vec{u}_1 \cdot \vec{u}_1) + v_2 (\vec{u}_1 \cdot \vec{u}_2) \\ (\vec{u}_2 \cdot \vec{V}) = v_1 (\vec{u}_2 \cdot \vec{u}_1) + v_2 (\vec{u}_2 \cdot \vec{u}_2) \end{cases}$$

where $$(\vec{u}_i \cdot \vec{V}) = \sum_{\alpha=(x,y,z)} u_{i\alpha} V_\alpha \quad -(9)$$

If we know $\vec{V}$, $\vec{u}_1$ & $\vec{u}_2$, then we can determine v_1 & v_2 by solving the simultaneous equations implied in (8).

But this was enabled by the notion of a dot-product. Can we define a similar quantity in our abstract vector space?

Turns out we can. This gives us the notion of <u>Inner Product spaces</u>.