18|8|2025 Day 7. Inner product and Dual spaces

Given a (complex) vector space V^N, we define an inner product rule for any 2 vectors $|v\rangle, |w\rangle \in V^N$, denoted as $\langle v|w\rangle$, such that

a) $\langle v|w\rangle \in \mathbb{C}$ (mapping to scalars)

b) $\langle v|w\rangle = \langle w|v\rangle^*$ (skew-symmetry)

$* \to$ complex conjugate

c) $\langle v|aw\rangle = a\langle v|w\rangle$ (linear)

$\langle av|w\rangle = a^*\langle v|w\rangle$ (antilinear)

d) $\langle v|\left(a|u\rangle + b|w\rangle\right)$

$= a\langle v|u\rangle + b\langle v|w\rangle$

(distributivity over vector addition)

e) $\langle v|v\rangle \geq 0$ (non-negativity for measure)

Property (a) defines a functional map from V^N to set of complex numbers $\mathbb{C}$. Note that we choose 2 vectors from V^N to make this map. (Binary operation)

$$\underset{\in V^N}{|v\rangle, |w\rangle} \longrightarrow \langle v|w\rangle \in \mathbb{C}$$

Property (b) says that the order in which we combine the 2 vectors matters and the results are related through a complex conjugation.

(Remember is $z = a + ib$, $a, b \in \mathbb{R}$ is a complex number then

$$z^* = a - ib$$

Or, in polar notation, if $z = re^{i\theta}$ then $z^* = re^{-i\theta}$ $r, \theta \in \mathbb{R}$.)

Property (e) allows us to associate a meaning to the inner-product of vector with itself. We say that, for a vector $|v\rangle \in V^N$,

$$\| \, |v\rangle \|^2 = \langle v|v\rangle \qquad — (10)$$

$\Leftrightarrow$ square of the norm (or length) of the vector $|v\rangle$.

So that the norm is

$$\boxed{\| \, |v\rangle \| = \sqrt{\langle v|v\rangle}} \qquad — (11)$$

<u>Theorem 2</u>: If, for a vector $|x\rangle \in V^N$,

$\langle x|x\rangle = 0$ then $|x\rangle \equiv |\emptyset\rangle$.

Conversely, the norm of the null vector is exactly zero.

A vector space for which such an inner product can be (and, hence, is) defined is called an <u>Inner Product Space</u>.

Note that while we defined the properties desirable of an inner (scalar) product, we haven't actually given a recipe to compute it. This, in turns out, depends on the vector space in question.

Eg. Consider the space formed by all doubles $\{(a, b), \text{ where } a, b \in \mathbb{R}\}$ over a field of real numbers.
Let's define an inner product as

$$\langle v | w \rangle \equiv a_v a_w + b_v b_w$$

$$\text{where} \quad |v\rangle = (a_v, b_v)$$
$$|w\rangle = (a_w, b_w)$$

a) Is this a vector space? b) Is this a valid inner product?

Solving the simultaneous equations in 13 will yield the components.

Dual Space and "bra" vectors

There's a more elegant (and formal) way to arrive at the notion of inner products. We note that an inner product is essentially a linear map from a vector space V^N to complex numbers $\mathbb{C}$. Such a map is termed as a (linear) functional.

For every $|v\rangle, |w\rangle \in V^N$ we define a functional

$$f_{|w\rangle}(|v\rangle) : V^N \longrightarrow \mathbb{C}$$

such that,

Linear in $|v\rangle$ i, $f_{|w\rangle}(a|v\rangle) = a f_{|w\rangle}(|v\rangle)$, $a \in \mathbb{C}$

anti Linear in $|w\rangle$ ii, $f_{a|w\rangle}(|v\rangle) = a^* f_{|w\rangle}(|v\rangle)$

(iii) $f_{|w\rangle}(|u\rangle + |v\rangle) = f_{|w\rangle}(|u\rangle) + f_{|w\rangle}(|v\rangle)$

Linearity over ket addition

(iv) $f_{|w\rangle+|x\rangle}(|v\rangle) = f_{|w\rangle}(|v\rangle) + f_{|x\rangle}(|v\rangle)$

Linearity over "bra" addition

(v) $f_{|w\rangle}(|v\rangle) = f^*_{|v\rangle}(|w\rangle)$

(vi) $f_{|v\rangle}(|v\rangle) \geq 0$

(vii) $f_{|0\rangle}(|v\rangle) = 0 \quad \forall\ |v\rangle \in V^N$ (14)

$\hookrightarrow$ null vector

Since each $f_{|w\rangle}$ depends on a $|w\rangle \in V^N$, we can assume an easier notation $\langle w|$, such that

$$\boxed{f_{|w\rangle}(|u\rangle) \equiv \langle w|v\rangle \quad \forall\ |v\rangle \in V^N} \quad (15)$$

The complete set of maps $\{\langle w| : |w\rangle \in V^N\}$ is easily shown to form a vector space. This is called the "Dual Space" of V^N, denoted as $\tilde{V}^N$.

Each vector $\langle v| \in \tilde{V}^N$ is called a "bra" vector. Rewriting all the properties in (14) in terms of the bra vectors we have:

i, $\langle w|\left(a|v\rangle\right) = a\langle w|v\rangle$ $\quad |v\rangle \in V^N$, $\langle w| \in \tilde{V}^N$, $a \in \mathbb{C}$

ii, $\langle aw|v\rangle = a^* \langle w|v\rangle$

iii, $\langle w|\left(|u\rangle + |v\rangle\right) = \langle w|u\rangle + \langle w|v\rangle$

iv, $\left(\langle w| + \langle x|\right)\left(|v\rangle\right) = \langle w|v\rangle + \langle x|v\rangle$

v, $\langle w|v\rangle^* = \langle v|w\rangle$

vi, $\langle v|v\rangle \geq 0$

vii, $\langle \emptyset|v\rangle = \langle v|\emptyset\rangle \equiv 0 \quad \forall |v\rangle \in V^N$

(15)

These are exactly the properties we required of our inner products. So, we can identify I.P.s as the combination of a vector each from $\mathbb{V}^N$ & its dual $\tilde{\mathbb{V}}^N$ such that the rules in 16 are obeyed.

Orthogonality: For $|v\rangle, |w\rangle \in \mathbb{V}^N$, if $\langle v|w\rangle$ (is defined and) is equal to 0, then the 2 vectors are said to be orthogonal.

$$\boxed{\langle v|w\rangle = 0} \qquad (17)$$

Alternatively, a vector $|w\rangle$ and a dual $\langle v|$ are said to be orthogonal if the condition in 17 is obeyed. Same is also true for $\langle w|$ & $|v\rangle$.

As mentioned before, given its non-negativity, we can associate $\langle V|V\rangle$ with the square norm/length of $|V\rangle$ i.e. $\||V\rangle\| = \sqrt{\langle V|V\rangle}$.

So we can define a unit vector as one with a unit length.
If $\langle u|u\rangle = 1 \Rightarrow |u\rangle$ is unit vector.

Given a vector $|V\rangle \in \mathbb{V}^N$, we can always define a unit vector $|v\rangle$ which is "parallel" to $|V\rangle$ by

$$|v\rangle \equiv \frac{|V\rangle}{\||V\rangle\|} = \frac{|V\rangle}{\sqrt{\langle V|V\rangle}} \quad (18)$$

This procedure is called Normalization.
In QM, only the "direction" of a ket vector decides the state. So states will often be normalized.