19/8/2025 Day 9. <u>Linear Operators</u>

<u>Orthonormal bases</u>: A set of linearly independent vectors $\mathbb{U} = \{|u_i\rangle, i = 1, N\}$ spanning V^N is said to form an orthogonal basis set if

$$\langle u_i | u_j \rangle \begin{cases} \neq 0 & \text{if } i = j \\ = 0 & \text{if } i \neq j \end{cases}$$

$$\forall\, i, j \in [1, N]. \quad (1)$$

If, additionally, the vectors are normalized we have the condition

$$\langle u_i | u_j \rangle = \delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases} \quad (2)$$

for an <u>orthonormal basis set</u>

Abbreviated as O/N.

By its definition, we can write any vector $V \in \mathcal{V}^N$ in terms $\mathrm{w.}(O/N)$

$$|V\rangle = \sum_{i=1}^{N} \alpha_i |u_i\rangle \quad — ③$$

where $\alpha_i \in \mathbb{C}$

α_i can be determined easily using ②.

$$\langle u_j | V\rangle = \sum_{i=1}^{N} \alpha_i \langle u_j | u_i\rangle$$

$$= \sum_{i=1}^{N} \alpha_i \delta_{ij} = \alpha_j \quad — ④$$

$$\therefore \quad \boxed{|V\rangle = \sum_{i=1}^{N} |u_i\rangle \underbrace{\langle u_i | V\rangle}_{\alpha_i}} \quad — ⑤$$

⑤ is one of the most important relations for QM. It says that any vector can be resolved into an O/N basis with coefficient obtained as inner products of the vector with the corresponding basis ket.

$$|v\rangle = \sum_{i=1}^{N} \underbrace{|u_i\rangle\langle u_i|}_{\hat{P}_i} v\rangle$$

The object $\hat{P}_i$ is interesting. Comparing with (5), its action can be given as:

$$\hat{P}_i |v\rangle = \alpha_i |u_i\rangle \quad \text{— (6)}$$

Thus, acting on $|v\rangle$ it generates the part of $|v\rangle$ parallel to $|u_i\rangle$. Both sides of (6) are vectors. Hence, $\hat{P}_i$ maps a vector $|v\rangle$ to another vector $\alpha_i |u_i\rangle$. Such an object is called an operator. In this case, we have a "projection" operator.

Note that, $\hat{P} \equiv \left(\sum_{i=1}^{N} \hat{P}_i\right)$ is also an operator. But this one maps $|v\rangle$ to $|v\rangle$

$$|v\rangle = \hat{P}|v\rangle \quad \text{— (7)}$$

Such an operator acts equivalent to a multiplication by 1. ∴ it is called a identity operator, denoted by $\hat{\mathbb{1}}$.

i.e. $$\boxed{\hat{P} = \sum_{i=1}^{N} P_i = \sum_{i=1}^{N} |u_i\rangle\langle u_i| = \hat{\mathbb{1}}} \quad (7)$$

This statement is called the <u>resolution of the identity</u> in the U basis.

The objects $|u_i\rangle\langle u_i|$ (or $\hat{P}_i$) are a combination of a bra and a ket but different from the inner product. This is because they map $V^N \to V^N$, whereas I.P.s map $V^N \to \mathbb{C}$.

These objects are called "Outer Products"

Hilbert spaces: Inner product spaces which are also a "complete metric space" are called Hilbert spaces.
Without worrying about the meaning of this definition, we will claim that the I.P. spaces of interest to us form Hilbert spaces. They will be denoted henceforth as $\mathcal{H}$.
Typically, the H.S. of our interest will be infinite dimensional ($N \to \infty$).
This does not present any conceptual difficulty.

We also note here that, usually, an O/N basis set is chosen to "represent" vectors in the H.S.

$$|V\rangle = \sum_{i=1}^{\infty} |u_i\rangle\langle u_i|V\rangle \quad -(8)$$

Which one we choose depends on our convenience.

Linear Operators:

Let us formally introduce the notion of linear operators.

hat will indicate operator

An object $\hat{\alpha}$ is said to be a linear operator on $\mathcal{H}$, if

(1) acting on any vector $|v\rangle \in \mathcal{H}$, it produces another vector (say) $|w\rangle \in V^N$.

i.e. $\hat{\alpha}|v\rangle = |w\rangle$ (9)

or $\hat{\alpha} : \mathcal{H} \longrightarrow \mathcal{H}$.

(2) $$\hat{\alpha}\,(c|v\rangle) = c\,\hat{\alpha}|v\rangle \quad \forall c \in \mathbb{C}$$

(3) $$\hat{\alpha}\,(|v\rangle + |u\rangle) = \hat{\alpha}|v\rangle + \hat{\alpha}|u\rangle$$

We can see that the projection operators defined earlier clearly meet this description. Operators can be of many kinds. Some common ones are:

1. Constant operators : $\hat{\alpha}\,|v\rangle = c\,|v\rangle$

operator

$c \in \mathbb{C}$

$\forall\, |v\rangle \in \mathcal{H}$

2. Identity Operator : Same definition as above but with $c \equiv 1$. In such a case we use the symbol $\hat{\mathbb{1}}$ for the operator.

3. Null operator : If $\hat{\alpha}\,|v\rangle = \phi \;\forall\, |v\rangle \in \mathcal{H}$ then $\hat{\alpha}$ is called a null operator.

In many ways operators can be thought of as numbers with some exception.

Addition of operators

(9.5) — $(\hat{\alpha} + \hat{\beta})\,|v\rangle \equiv \hat{\alpha}\,|v\rangle + \hat{\beta}\,|v\rangle$

$\in \mathcal{H}$

Since the result of the addition operation is also a vector, we can think of $\hat{\alpha} + \hat{\beta}$ as some operator $\hat{\gamma}$, such that

$$\hat{\gamma}\,|v\rangle \equiv (\hat{\alpha}+\hat{\beta})|v\rangle = \hat{\alpha}|v\rangle + \hat{\beta}|v\rangle \quad (10)$$

This lets us define the operator identity

$$\hat{\gamma} \equiv \hat{\alpha} + \hat{\beta} \quad (11)$$

[(10) $\Longleftrightarrow$ (11)]

<u>Scaling operator</u>: If $a \in \mathbb{F}$, $|v\rangle \in \mathcal{H}$. then

$$(a\hat{\alpha})|v\rangle = a(\hat{\alpha}|v\rangle) = \hat{\alpha}\,|av\rangle \quad (12)$$

<u>Product of operators</u>:

$$(\hat{\alpha}\hat{\beta})|v\rangle \equiv \hat{\alpha}\,(\hat{\beta}|v\rangle) \quad (13)$$

$\forall\, |v\rangle \in \mathcal{H}$.

If there exists an operator $\hat{\gamma}$ such that

$$\hat{\gamma}\,|v\rangle = (\hat{\alpha}\hat{\beta})\,|v\rangle \;\; \forall\, |v\rangle \in \mathcal{H} \quad (14)$$

then we can write,

$$\hat{\gamma} = \hat{\alpha}\hat{\beta} \quad (15)$$

(14) $\Longleftrightarrow$ (15)

Note, however, that

$$\hat{\alpha}\hat{\beta} \neq \hat{\beta}\hat{\alpha} \quad \text{— (16)}$$

in general.

Thus, operators do not "commute" in general.

Through these equations we have defined an operator algebra.

Operator identity: If $\hat{\alpha}|v\rangle = \hat{\beta}|v\rangle$ $\forall\, |v\rangle \in \mathcal{H}$

then $$\boxed{\hat{\alpha} = \hat{\beta}} \quad \text{— (17)}$$