

20/8/2025

Day 10. Hermilian Operators

Action on duals: The same operators can also be understood to act on dual vectors to yield other dual vectors. Here's how:

Let $\hat{a}|v\rangle = |w\rangle$ for some $|w\rangle \in \mathcal{H}$ — (1)
then

$$\langle u|\hat{a}|v\rangle \equiv \langle u|w\rangle \quad \text{--- (2)}$$

Since $|v\rangle$ can be any arbitrary vector and $|w\rangle$ its image in $\mathcal{H}$ under $\hat{a}$, we can say that,

$$\langle x| \equiv \langle u|\hat{a} \quad \text{--- (3)}$$

is that dual vector ($\in \mathcal{H}^*$) which would yield the same value for inner product with $|v\rangle$ as.

$\langle u|$ would with $|w\rangle$. That is,

$$\langle x|: \quad \langle x|v\rangle = \langle u|w\rangle \quad \text{--- (4)}$$

$\equiv \langle u|\hat{a}$

This defines the action of $\hat{a}$ on a bra vector (from the right side).

Postulate 2: Every dynamical variable of a (quantum) system, at each instant, corresponds to a linear operator on the Hilbert space of quantum states.

Real dynamical variables will be seen to correspond to real or Hermitian operators.

By dynamical variables we mean all mechanical (and other) properties that describe all aspects of the instantaneous state of a system. These include positions, momenta, velocities, angular momenta, dipole moments, charge densities, magnetic moments, etc.

Operator will be denoted by a hat appearing on top of the usual property symbol.
e.g. $\vec{p} \rightarrow \hat{\vec{p}}$, $\vec{\mu} \rightarrow \hat{\vec{\mu}}$, $x \rightarrow \hat{x}$,
etc.

Adjoint of an operator : We defined above $\langle x | \equiv \langle u | \hat{\alpha}$.

$\langle x |$ itself is the dual of $|x\rangle$ which can be, in principle, connected to the ket $|u\rangle$ via some operator.

$$\text{Let } |x\rangle = \hat{\beta} |u\rangle \quad \text{--- (5)}$$

Is there a connection between $\hat{\alpha}$ & $\hat{\beta}$?
Yes. $\hat{\beta}$ is called the Hermitian adjoint of $\hat{\alpha}$ and is denoted by $\hat{\alpha}^\dagger$. Thus,

$$\text{if } \langle x | \equiv \langle u | \hat{\alpha}$$

$$\text{then } \boxed{|x\rangle = \hat{\alpha}^\dagger |u\rangle} \quad \text{--- (6)}$$

From (5) we have,

$$\langle x | v \rangle = \langle u | \hat{\alpha} | v \rangle$$

$$= \langle v | x \rangle^* = \langle v | \hat{\alpha}^\dagger | u \rangle^* \quad \text{--- (7)}$$

$$\therefore \boxed{\langle u | \hat{\alpha} | v \rangle^* = \langle v | \hat{\alpha}^\dagger | u \rangle} \quad \text{--- (8)}$$

⑧ is an alternative definition for $\hat{\alpha}^\dagger$.
 Quantities of the kind $\langle u | \hat{\alpha} | v \rangle$ are
 called matrix elements. We will
 see later why.

Let $\hat{\beta}^\dagger = \hat{\alpha}$. Then, ⑧ implies

$$\langle v | (\hat{\beta}^\dagger)^\dagger | u \rangle = \langle u | \hat{\beta}^\dagger | v \rangle^* \\
= \langle v | \hat{\beta} | u \rangle \quad \text{--- ⑨}$$

Since this is true for any $|u\rangle, |v\rangle \in \mathcal{H}$,
 we have

$$\boxed{(\hat{\beta}^\dagger)^\dagger = \hat{\beta}} \quad \text{--- ⑩}$$

[Logical flow of the arguments is define the adjoint

$$\begin{array}{ccc} |x\rangle & \xrightarrow{f} & \langle x| = \langle u | \hat{\alpha} \\ \uparrow ? & & \uparrow \hat{\alpha}^\dagger \\ |u\rangle & \xrightarrow{f} & \langle u| \\ \Downarrow & & \\ |x\rangle \equiv \alpha^\dagger |u\rangle & & \end{array}$$

That is, the Hermitian adjoint of the Hermitian adjoint of an operator is the operator itself.

This is similar to $(c^*)^* = c \in \mathbb{C}$. Thus, operator algebra resembles that of complex numbers with the caveat that $\hat{\alpha}\hat{\beta} \neq \hat{\beta}\hat{\alpha}$, in general.

Postulate 2 then implies if a dynamical variable ξ is associated with the operator $\hat{\xi}$ ('xi') then its complex conjugate is associated with $\hat{\xi}^\dagger$.

$$\left. \begin{aligned} \xi &\Leftrightarrow \hat{\xi} \\ \xi^* &\Leftrightarrow \hat{\xi}^\dagger \end{aligned} \right\} \textcircled{9.5}$$

Hence for real dynamical variables α , where $\alpha = \alpha^*$, we must have, for the corresponding operator $\hat{\alpha}$, that:

$$\boxed{\hat{\alpha} = \hat{\alpha}^\dagger} \text{ --- } \textcircled{10}$$

Such operators are termed Hermitian or self-adjoint or "real".

We can also include this assertion in postulate 2 but it is not necessary since the requirement follows more intuitively.

Note that given any operator $\hat{a}$ on $\mathcal{H}$, we can write.

$$\begin{aligned}\hat{a} &= (\hat{a} + \hat{a}^\dagger) + (\hat{a} - \hat{a}^\dagger) \\ &\equiv \hat{a}_R + \hat{a}_I \quad - (11)\end{aligned}$$

where $\hat{a}_R = \hat{a} + \hat{a}^\dagger \Rightarrow \hat{a}_R^\dagger = \hat{a}_R \rightarrow$ Hermitian (real)

$\hat{a}_I = \hat{a} - \hat{a}^\dagger \Rightarrow \hat{a}_I^\dagger = -\hat{a}_I \rightarrow$ skew-Hermitian (imaginary)

So that $\hat{a}^\dagger = \hat{a}_R - \hat{a}_I \quad - (12)$

Any operator can, hence, be written as a sum of a real (Hermitian) and an imaginary (skew-Hermitian) operators, just like a complex number.

Properties of adjoints (Prove as H.W.)

$$1. (\hat{\alpha} + \hat{\beta})^\dagger = \hat{\alpha}^\dagger + \hat{\beta}^\dagger$$

$$2. (\hat{\alpha}\hat{\beta})^\dagger = \hat{\beta}^\dagger \hat{\alpha}^\dagger$$

$$3. (\hat{\alpha}\hat{\beta}\hat{\gamma}\dots\hat{\delta})^\dagger = \hat{\delta}^\dagger \dots \hat{\gamma}^\dagger \hat{\beta}^\dagger \hat{\alpha}^\dagger$$

$$4. (|v\rangle\langle u|)^\dagger = |u\rangle\langle v|$$

$$5. \boxed{\langle u|\hat{\alpha}|v\rangle^* = \langle v|\hat{\alpha}|u\rangle} \text{ for a real operator}$$

Alternative definition for Hermitian operators.

$$6. \langle v|\hat{\alpha}|v\rangle \in \mathbb{R} \quad \forall |v\rangle \in \mathcal{H} \text{ for a real operator } \hat{\alpha}.$$

$$7. \text{ If } \hat{\alpha} \text{ is real (Hermitian) } i\hat{\alpha} \text{ is purely imaginary (skew-Hermitian)}$$

Eigenvalue equations: Given a linear operator $\hat{\alpha}$, it is possible to ask for vectors $\{|v_\alpha\rangle\}$ such that when $\hat{\alpha}$ acts on them they merely get scaled: i.e.

$$\hat{\alpha} |v_\alpha\rangle = c_\alpha |v_\alpha\rangle \quad - (14)$$

$\searrow \in \mathbb{C}$

Such an equation is called the eigenvalue equation of $\hat{\alpha}$. $|v_\alpha\rangle$ is called an eigenvector and the corresponding scalar c_α is called the eigenvalue.

Hermitian operators have interesting properties associated with their eigenvalue equations.

Let $\hat{\alpha}$ be a Hermitian operator.
 $\Rightarrow \hat{\alpha}^\dagger = \hat{\alpha}$.

Let the corresponding eigenvalue equation be:

$$\hat{\alpha} |v_{\alpha}\rangle = c_{\alpha} |v_{\alpha}\rangle \quad - (15)$$

$$\Rightarrow \langle v_{\alpha} | \hat{\alpha} | v_{\alpha} \rangle = c_{\alpha} \langle v_{\alpha} | v_{\alpha} \rangle \quad - (16)$$

The property 6. in eq. (13) implies that

$\langle v_{\alpha} | \hat{\alpha} | v_{\alpha} \rangle$ is real

Since $\langle v_{\alpha} | v_{\alpha} \rangle \geq 0$ and real, this implies c_{α} is real. So we get a very important result.

The eigenvalues of a Hermitian operator are real.

Consider 2 different eigenkets $|v_{\alpha}\rangle, |v_{\alpha'}\rangle$

$$\hat{\alpha} |v_{\alpha}\rangle = c_{\alpha} |v_{\alpha}\rangle \quad - (17)$$

$$\hat{\alpha} |v_{\alpha'}\rangle = c_{\alpha'} |v_{\alpha'}\rangle \quad - (18)$$

Taking inner product of (15) with $\langle v_{\alpha'} |$ we get

$$\langle v_{\alpha'} | \hat{\alpha} | v_{\alpha} \rangle = c_{\alpha} \langle v_{\alpha} | v_{\alpha'} \rangle \quad - (19)$$

Similarly, taking inner product of (18) with $\langle v_{\alpha} |$ and then taking the complex conjugate of both sides yields.

$$\langle v_{\alpha} | \hat{\alpha} | v_{\alpha'} \rangle^* = c_{\alpha'} \langle v_{\alpha} | v_{\alpha'} \rangle^* \quad - (20)$$

$\Rightarrow$ (using property 5. in eq (13))

$$\langle v_{\alpha'} | \hat{\alpha} | v_{\alpha} \rangle = c_{\alpha'} \langle v_{\alpha'} | v_{\alpha} \rangle \quad - (21)$$

(19) - (20) gives:

$$(c_{\alpha} - c_{\alpha'}) \langle v_{\alpha'} | v_{\alpha} \rangle = 0 \quad - (22)$$

We consider 2 cases below.

i) $C_\alpha \neq C_{\alpha'}$ distinct eigenvalues

Then,

$$\langle V_{\alpha'} | V_\alpha \rangle = 0 \quad - (23)$$

i.e. Eigenvectors corresponding to 2 distinct eigenvalues of a Hermitian operator are orthogonal.

ii) $C_\alpha = C_{\alpha'}$ (but $\alpha \neq \alpha'$ necessarily)

Such a case is referred to as a degeneracy & the eigenvalue is said to be degenerate. $|V_\alpha\rangle$ & $|V_{\alpha'}\rangle$ are not the same (or parallel).

Note that any combination of $|V_\alpha\rangle$ & $|V_{\alpha'}\rangle$ also is an eigenvector $\hat{\alpha}$.

$$\hat{\alpha} (a|V_\alpha\rangle + b|V_{\alpha'}\rangle) = C_\alpha (a|V_\alpha\rangle + b|V_{\alpha'}\rangle) \quad - (24)$$

We can then construct 2 combinations in the following way.

$$\begin{aligned} \text{def } |u_1\rangle &\equiv |v_\alpha\rangle / \sqrt{\langle v_\alpha | v_\alpha \rangle} \\ \text{and } |u_2\rangle &\equiv |v_{\alpha'}\rangle / \sqrt{\langle v_{\alpha'} | v_{\alpha'} \rangle} \end{aligned} \quad \} \textcircled{25}$$

These are the unit vectors along $|v_\alpha\rangle$ & $|v_{\alpha'}\rangle$.
We can choose one of the eigenvectors to be $|u_1\rangle$ itself. i.e.

$$|e_1\rangle = |u_1\rangle \quad - \textcircled{26}$$

The second eigenvector orthogonal to $|e_1\rangle$ is written as:

$$|E\rangle = |u_2\rangle - |e_1\rangle \langle e_1 | u_2 \rangle \quad - \textcircled{27}$$

It is easy to check that this is orthogonal to $|e_1\rangle$.

$$\begin{aligned} \langle e_1 | E \rangle &= \langle e_1 | u_2 \rangle - \langle e_1 | e_1 \rangle \langle e_1 | u_2 \rangle \\ &= \langle e_1 | u_2 \rangle - \langle e_1 | u_2 \rangle = 0 \\ &\quad (\because |e_1\rangle \text{ is normalized}) \end{aligned} \quad - \textcircled{28}$$

$$\therefore |e_2\rangle = \frac{|E\rangle}{\sqrt{\langle E|E\rangle}} \quad \text{--- (29)}$$

is the 2nd o/n eigenvector.

Hence,

Eigenvectors of a Hermitian are mutually orthogonal or can be chosen to be orthogonal (for degenerate cases).

To establish further the connection between Hermitian Operators and quantum mechanics we next set up the "measurement" postulates.

The Measurement Postulates

Postulate 3 (a)

Any result of measurement of a real dynamical variable is one of the eigenvalues of the corresponding operator. Conversely, every eigenvalue is a possible result of a measurement of the dynamical variable for some state of the system.

→ Physical meaning of eigenvalues of a Hermitian Operator.

⇒ all possible values can be measured and correspond to the eigenvalues of $\hat{\alpha}$

Postulate 3(b):

If the system is in a dynamical state corresponding to the eigenket of a real operator $\hat{A}$, belonging to the eigenvalue α' , then a measurement of $\hat{A}$ will certainly yield the value α' .

Conversely, if the system is in a state such that a measurement of $\hat{A}$ is certain to yield the value α' , then the state is an eigenstate of $\hat{A}$.

N.B. eigenket $\leftrightarrow$ eigenstate

Implications:

- (1) α' degenerate $\Rightarrow$ state is in superposition of degenerate eigenstates.
(Resolution incomplete)
- (2) $\because$ 2 states of differing eigenvalues are orthogonal, if 2 preparations are certain to yield different values of $\hat{A}$, they are

orthogonal

→ Physical meaning of orthogonality

After a measurement yielding the eigenvalue α' , the system, being perturbed by the act of measuring "jumps" into the eigenstate $|\alpha'\rangle$.

Subsequent measurements continue to yield the value α' confirming this understanding.

Postulate 3(c): Collapse postulate

The state of a system depends linearly on the states that it "jumps" into following a measurement.

Since 3(a) says all eigenvalues of a real variable are measurable and 3(b) states the resulting states are eigenstates, what we are saying above is that any state depends linearly on the eigenstates of a real variable.

Complete set of states: A set of states on which any vector in space depends and hence can be used to fully represent any state.

Not all Hermitian operators have enough eigenvectors to satisfy this requirement.

Only those that do can be observed. Hence, there are called observables. For these,

$$|\psi\rangle = \sum_{\alpha'} c_{\alpha'} |\alpha'\rangle \quad (32)$$