1 Tensor Product states

Let V_1 and V_2 be two vector spaces with $|\psi\rangle \in V_1$ and $|\phi\rangle \in V_2$. We define a tensor product of V_1 and V_2 as

$$V = V_1 \otimes V_2 \tag{1}$$

such that

$$|\psi\rangle_1 \otimes |\phi\rangle_2 \in V \tag{2}$$

In equation 2, the subscripts indicate the original vector space the state is from.
It can be shown that V forms a vector space if

1. $$a\left(|\psi\rangle_1 \otimes |\phi\rangle_2\right) = \left(a\,|\psi\rangle\right) \otimes |\phi\rangle_2 = |psi\rangle_1 \otimes \left(a\,|phi\rangle_2\right) \tag{3}$$

   for $a \in \mathbb{C}$. i.e. it is linear over scalar multiplication.

2. $$\left(|\psi_1\rangle_1 + |\psi_2\rangle\right) \otimes |\phi\rangle_2 = |\psi_1\rangle_1 \otimes |\phi\rangle_2 + |\psi_2\rangle_1 \otimes |\phi\rangle_2 \tag{4}$$

   i.e. distributive over vector addition.

In equation 4, the resulting vector is also in V to ensure closure under addition.

2 Basis states in tensor product spaces

If $\{|u_i\rangle\}$ and $\{|v_j\rangle\}$ are sets of orthonormal bases for V_1 and V_2, respectively, then $\left\{|u_i\rangle_1 \otimes |v_j\rangle_2\right\}$ forms an orthonormal basis set for $V = V_1 \otimes V_2$. Consider a state $|\psi\rangle_1 \otimes |\phi\rangle_2 \in V$. Then, using the definition of the basis sets of the individual spaces, we can write

$$\begin{aligned} |\psi\rangle_1 \otimes |\phi\rangle_2 &= \left(\sum_i c_i\,|u_i\rangle_1\right) \otimes \left(\sum_j d_j\,|v_j\rangle_2\right) \\ &= \sum_{i,j} c_i d_j\,|\psi\rangle_1 \otimes |\phi\rangle_2 \end{aligned} \tag{5}$$

i.e. vectors in V can be expanded in terms of $\left\{|u_i\rangle_1 \otimes |v_j\rangle_2\right\}$.
Thus any general $|\Psi\rangle \in V$ can be written as

$$|\Psi\rangle = \sum_{i,j} a_{i,j}\,|u_i\rangle_1 \otimes |v_j\rangle_2 \tag{6}$$

However, it is not always necessary that this can be "factored" into the tensor product of two vectors from V_1 and V_2. i.e. in general,

$$|\Psi\rangle \neq |\psi\rangle_1 \otimes |\phi\rangle_2 \tag{7}$$

for any $|\psi\rangle \in V_1$ and any $|\phi\rangle \in V_2$. States that cannot be "factored" in this way are entangled in quantum mechanics.

3 Inner Products

Consider $|\psi\rangle, |\psi'\rangle \in V_1$ and $|\phi\rangle, |\phi'\rangle \in V_2$. This implies $|\psi\rangle_1 \otimes |\phi\rangle_2, |\psi'\rangle_1 \otimes |\phi'\rangle_2 \in V$. The inner products in the Tensor Product spaces are defined as

$$\left({}_1\langle\psi| \otimes {}_2\langle\phi|\right) \cdot \left(|\psi'\rangle_1 \otimes |\phi'\rangle_2\right) = {}_1\langle\psi|\psi'\rangle_1\ {}_2\langle\phi|\phi'\rangle_2 \tag{8}$$

This definition can also be used to show that the basis states used in equation 6 are also orthonormal if the original bases in V_1 and V_2 are orthonormal.

4 Operators

Consider $\hat{A}_1$ on V_1, $\hat{B}_2$ on V_2 we define

$$\left(\hat{A}_1 \otimes \hat{B}_2\right)\left(|\psi\rangle_1 \otimes |\phi\rangle_2\right) = \left(\hat{A}_1 |\psi\rangle_1\right) \otimes \left(\hat{B}_2 |\phi\rangle_2\right) \tag{9}$$

Consider the general state given in equation 6. Then,

$$\left(\hat{A}_2 \otimes \hat{B}_2\right)|\Psi\rangle = \sum_{i,j} a_{i,j} \left(\hat{A}_1 |u_i\rangle_1\right) \otimes \left(\hat{B}_2 |v_j\rangle_2\right) \tag{10}$$

Suppose we want to define an operator, $\hat{A}$, that acts only on the V_1 part of the state $|\Psi\rangle \in V$. We can define it as $\hat{A}_1 \otimes \hat{\mathbb{1}}_2$, or simply write it as $\hat{A}_1$. We will often use the latter notation for simplicity. We interpret its action as below.

$$\begin{aligned}\hat{A}_1 \otimes \hat{\mathbb{1}}_2 |u_i\rangle_1 \otimes |v_j\rangle_2 &= \left(\hat{A}_1 |u_i\rangle_1\right) \otimes \left(\hat{\mathbb{1}}_2 |v_j\rangle_2\right) \\ &= \left(\hat{A}_1 |u_i\rangle_1\right) \otimes |v_j\rangle_2 \end{aligned} \tag{11}$$

A similar definition also holds for $\hat{B}_2 = \hat{\mathbb{1}}_2 \otimes \hat{B}_2$.
The following notations are often used equivalently.

$$|\psi\rangle_1 \otimes |\phi\rangle_2 \longrightarrow |\psi\rangle_1 |\phi\rangle_2 \longrightarrow |\psi, \phi\rangle \tag{12}$$

$$\tag{13}$$

Examples of such states are

1. Position kets of a particle in 3-dimensions – $|\mathbf{r}\rangle = |x\rangle \otimes |y\rangle \otimes |z\rangle$.
2. A spin-orbital state of an electron – $|\psi_{n,m}\rangle = |\phi_n(\mathbf{r})\rangle \otimes |\chi_m(\sigma)\rangle$

Also,

$$\hat{A}_1 \otimes \hat{B}_2 \longrightarrow \hat{A}_1 \hat{B}_2 \tag{14}$$

Examples of such operators are

1. Kinetic energy operator of particle in 3-dimensions– $\hat{T} = \frac{\hat{p}_1^2}{2} + \frac{\hat{p}_2^2}{2} + \frac{\hat{p}_3^2}{2}$.
2. Spin-orbit coupling operator $\hat{H}_{SO} = \zeta \hat{\vec{L}} \cdot \hat{\vec{S}}$ (orbital and spin angular momentum operators act on different spaces).

5 Permutations

5.1 Permutation operators

Permutations are defined as rearrangements or elements in a set. We define such an operator's action on tensor product states as

$$\hat{P}_{mnp} |u_i\rangle_1 |u_j\rangle_2 |u_k\rangle_3 = |u_i\rangle_m |u_j\rangle_n |u_k\rangle_p \tag{15}$$

where $m \neq n \neq p$ and they borrow values from $1, 2, 3$. For example,

$$\hat{P}_{231} |u_i\rangle_1 |u_j\rangle_2 |u_k\rangle_3 = |u_i\rangle_2 |u_j\rangle_3 |u_k\rangle_1 \tag{16}$$

The permutation could equivalently be performed on the states, keeping the space index in the original order. That is,

$$\hat{P}_{231} |u_i\rangle_1 |u_j\rangle_2 |u_k\rangle_3 = |u_k\rangle_1 |u_i\rangle_2 |u_j\rangle_3 \tag{17}$$

Both definitions are equivalent and represent the freedom in ordering spaces in a tensor product. In what follows, we will generically denote permutations as $\hat{P}_\alpha$. Note that a set of N items can be permuted in $N!$ ways. Also note that $P_{123..n} = \hat{\mathbb{1}}$, as it leaves the original order unchanged.

5.2 Transpositions

Permutations that exchange 2 items, leaving the others unchanged, are termed transpositions. E.g., for $N = 3$, $\hat{P}_{132}, \hat{P}_{213}, \hat{P}_{321}$ are transpositions (or pair permutations).
Transpositions (denoted below as $\hat{T}_\nu$) can be shown to be satisfy

$$\hat{T}_\nu = \hat{T}_\nu^\dagger \quad \text{Hermitian} \tag{18}$$

$$\hat{T}_\nu^2 = \hat{\mathbb{1}} \quad \text{Involutory} \tag{19}$$

$$\hat{T}_\nu \hat{T}_\nu^\dagger = \hat{T}_\nu^\dagger \hat{T}_\nu = \hat{\mathbb{1}} \quad \text{Unitary} \tag{20}$$

Q. Prove the above relations.
Theorem 1: Every permutation can be written as a product of transpositions. i.e. $\hat{P}_\alpha = \hat{T}_{\alpha_1} \hat{T}_{\alpha_2} \ldots \hat{T}_{\alpha_n}$.

An example for $N = 3$ is $\hat{P}_{312} = \hat{P}_{132}\hat{P}_{213}$. Note that the terms on the right of this equation are transpositions. However, there are multiple ways of decomposing a permutation into transpositions.

Theorem 2: The number of transpositions involved in achieving a permutation is always either even or odd and decides the parity of the permutation. A given permutation has a fixed parity, regardless of the choice of the sequence of transpositions made to achieve it.

In particular, we can assign a number η_α for every permutation α of a set, such that $\eta_\alpha = \pm 1$, where the plus sign indicates even and minus sign odd parity.
Using these theorems, the following maybe shown.

1. $\hat{P}_\alpha^{-1} = \hat{P}_\alpha$
2. A permutation and its hermitian adjoint have the same parity.
3. $\hat{P}_\alpha \neq \hat{P}_\alpha^\dagger$ in general.

It is straightforward to see that the set of all permutations of the tensor products form a (non-abelian) group. In particular, there is an identity element ($\hat{P}_{123..n}$), product of any two permutations also is a permutation (closure), and every permutation has an inverse which also exists in the same group. This group is called the *symmetric group*.

Rearrangement Theorem: Product of a given permutation $hatP_\alpha$ with each member of the symmetric group results in only a rearrangement of the members without repeating any one of them.
A special consequence of this is that

$$\hat{P}_\alpha \left(\sum_\beta \hat{P}_\beta \right) = \left(\sum_\beta \hat{P}_\beta \right) \tag{21}$$

Q. Prove this result.